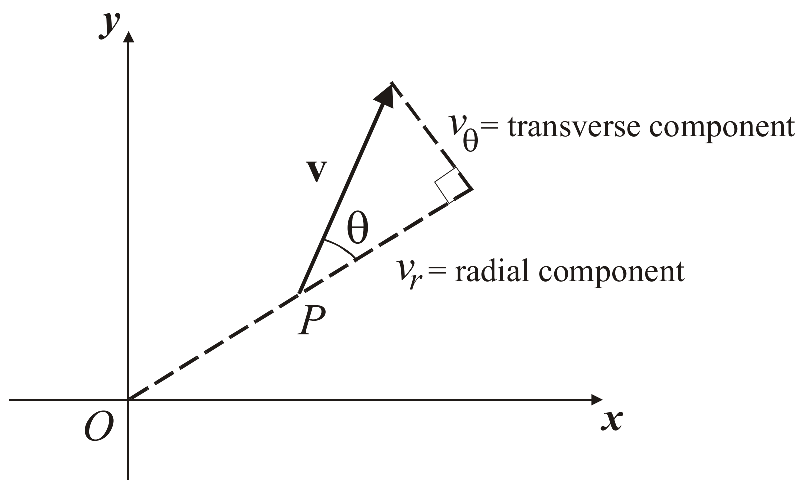
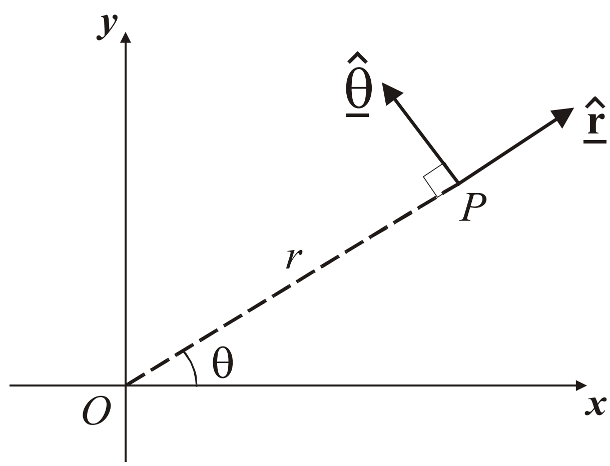
Motion described in polar coordinates

Vectors in polar co-ordinates

Suppose a particle at point $P = [r, \theta]$ - that is a particle whose position is given in polar co-ordinates - has velocity **v**. The velocity can be described in terms of its radical and the transverse components



In order to do so we define $\hat{\mathbf{r}}$ to be a unit vector in the radical direction and $\hat{\boldsymbol{\theta}}$ to be a unit vector in the transverse direction - that is, perpendicular to $\hat{\mathbf{r}}$ and taken in an anti- clockwise direction.



So the velocity is $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}$ where v_r, v_θ are the magnitudes of the radical and transverse components respectively.

The position of the particle at P can also be given in these polar vectors.
$\overrightarrow{OP} = \mathbf{r} = r\hat{\mathbf{r}}$

where r is the magnitude of the distance of P from the origin.

If an object is moving in a circle with centre O and radius a, then its position is

$\mathbf{r} = a\hat{\mathbf{r}}$

and its velocity is

$\underline{v} = a\omega\hat{\boldsymbol{\theta}}$

its acceleration is $\mathbf{a} = -a\omega^2\hat{\mathbf{r}}$ that is its acceleration is directed along the radius towards the centre.

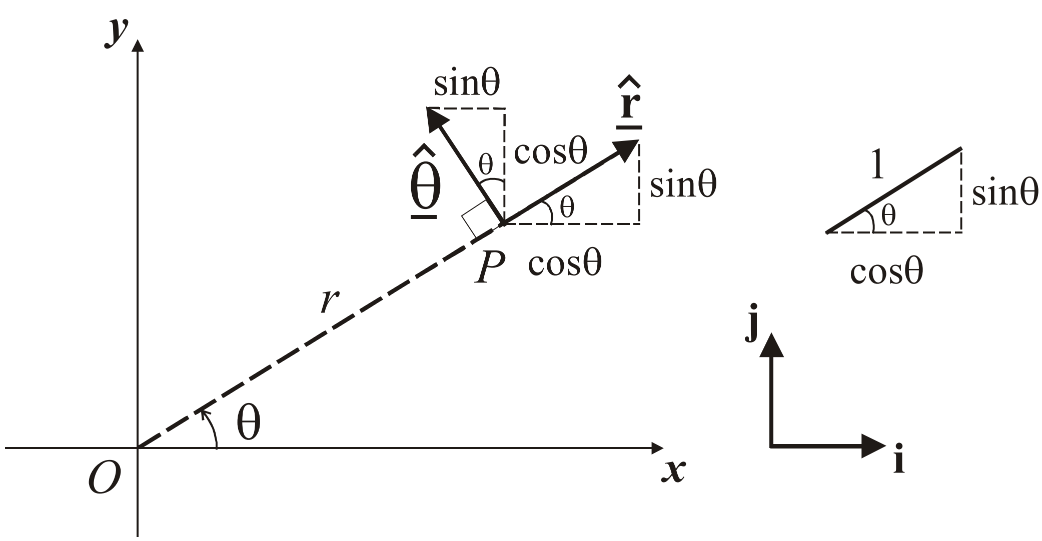
Change of basis

The unit vectors $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ represent fixed directions, but the vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ change direction as θ varies.

In two dimensions both sets of unit vectors- that is $\{\hat{\underline{\mathbf{i}}}, \hat{\underline{\mathbf{j}}}\}$ and $\{\hat{\underline{\mathbf{r}}}, \hat{\underline{\boldsymbol{\theta}}}\}$ form a basis- that is , any two dimensional vector $\hat{\underline{\mathbf{r}}}$ can be written in terms of either set

$\hat{\underline{\mathbf{r}}} = a\hat{\underline{\mathbf{i}}} + b\hat{\underline{\mathbf{j}}} = \alpha\hat{\underline{\mathbf{r}}} + \beta\hat{\underline{\boldsymbol{\theta}}}$

so it would be useful to know how to convert from one basis to the other.



As the diagram indicates,

$\underline{\mathbf{r}} = \cos\theta \underline{\mathbf{i}} + \sin\theta \hat{\underline{\mathbf{j}}}$

$\hat{\underline{\boldsymbol{\theta}}} = -\sin\theta \hat{\underline{\mathbf{i}}} + \cos\theta \underline{\mathbf{j}}$

This can be written in matrix form

$$\begin{pmatrix} \hat{\underline{\mathbf{r}}} \\ \hat{\underline{\boldsymbol{\theta}}} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\underline{\mathbf{i}}} \\ \hat{\underline{\mathbf{j}}} \end{pmatrix}$$

the matrix $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ represents a rotation through θ degrees. Its inverse is a rotation through $-\theta$ degrees.

$$\underline{\mathbf{R}}_\theta = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ can be given in terms of $\underline{\mathbf{r}}$ and $\hat{\underline{\boldsymbol{\theta}}}$ as

$$\begin{pmatrix} \underline{\mathbf{i}} \\ \underline{\mathbf{j}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{\underline{\mathbf{r}}} \\ \hat{\underline{\boldsymbol{\theta}}} \end{pmatrix}$$

$\hat{\underline{\mathbf{i}}} = \cos\theta \hat{\underline{\mathbf{r}}} - \sin\theta \hat{\underline{\boldsymbol{\theta}}}$

$\hat{\underline{\mathbf{j}}} = \sin\theta \hat{\underline{\mathbf{r}}} + \cos\theta \hat{\underline{\boldsymbol{\theta}}}$

Example

A particle has velocity

$\underline{\mathbf{v}} = 4\cos\theta \hat{\underline{\mathbf{r}}} - 4\sin\theta \hat{\underline{\boldsymbol{\theta}}}$

What is its velocity when referred to Cartesian co-ordinates?

Solution

Let the velocity in $\underline{\mathbf{i}}, \underline{\mathbf{j}}$ co-ordinates be

$\underline{\mathbf{v}} = \alpha \underline{\mathbf{i}} + \beta \hat{\underline{\mathbf{j}}}$

then

$\alpha \underline{\mathbf{i}} = \{\cos\theta\} v_r \underline{\hat{\mathbf{r}}} - \{\sin\theta\} v_\theta \underline{\hat{\boldsymbol{\theta}}}$

$\alpha = \cos\theta(4\cos\theta) - \sin\theta(-4\sin\theta) = 4\left(\cos^2\theta + \sin^2\theta\right) = 4$

and

$\beta \underline{\hat{\mathbf{j}}} = (-\sin\theta) v_r \underline{\hat{\mathbf{r}}} - (\cos\theta) v_\theta \underline{\hat{\boldsymbol{\theta}}}$
$\overline{\beta} = -\sin\theta(4\cos\theta) - \cos\theta(-4\sin\theta) = 0$

So $\underline{\mathbf{v}} = 4\underline{\mathbf{i}}$

This indicates that the velocity is constant in the $\underline{\mathbf{i}}$ direction.

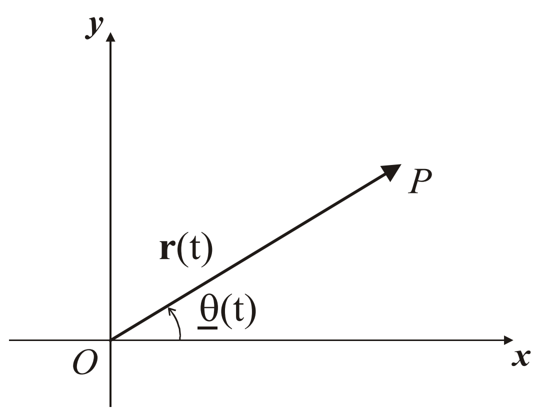
Rate of change of $\underline{\hat{\mathbf{r}}}$ and $\underline{\hat{\boldsymbol{\theta}}}$

A unit vector in the radial direction is

$\underline{\hat{\mathbf{r}}} = \cos\theta \underline{\mathbf{i}} + \sin\theta \underline{\mathbf{j}}$

Suppose $\underline{\hat{\mathbf{r}}}$ is in fact a function of another parameter t – for example t could represent time.

Then $\underline{\hat{\mathbf{r}}} = \underline{\hat{\mathbf{r}}}(t)$ would give the position of the radial vector at time t.



Then $\underline{\hat{\mathbf{r}}}(t) = \frac{d\underline{\hat{\mathbf{r}}}}{dt}$

Using the chain rule

$$\underline{\dot{\hat{\mathbf{r}}}}(t) = \frac{d\underline{\hat{\mathbf{r}}}}{d\theta}\frac{d\theta}{dt}$$

since $\hat{\mathbf{r}} = \cos\theta\underline{\mathbf{i}} + \sin\theta\underline{\mathbf{j}}$

then

$$\frac{d\underline{\hat{\mathbf{r}}}}{d\theta} = -\sin\theta\underline{\mathbf{i}} + \cos\theta\underline{\mathbf{j}}$$

so

$$\underline{\dot{\hat{\mathbf{r}}}}(t) = \left(-\sin\theta\underline{\mathbf{i}} + \cos\theta\underline{\mathbf{j}}\right)\frac{d\theta}{dt} = \underline{\hat{\boldsymbol{\theta}}}\ \frac{d\theta}{dt} = \dot{\theta}\underline{\hat{\boldsymbol{\theta}}}$$

where $\dot{\theta} = \dfrac{d\theta}{dt}$

The rate of change of $\underline{\hat{\mathbf{r}}}$ is in the $\underline{\hat{\boldsymbol{\theta}}}$ direction and has magnitude $\dot{\theta} = \dfrac{d\theta}{dt}$

We use the same approach to find the rate of change of $\underline{\hat{\boldsymbol{\theta}}}$.

$$\underline{\hat{\boldsymbol{\theta}}} = -\sin\theta\underline{\mathbf{i}} + \cos\theta\underline{\mathbf{j}}$$

$$\frac{d\underline{\hat{\boldsymbol{\theta}}}}{d\theta} = -\cos\theta\underline{\mathbf{i}} - \sin\theta\underline{\mathbf{j}}$$

Therefore

$$\begin{aligned}\frac{d\underline{\hat{\boldsymbol{\theta}}}}{dt} &= \frac{d\underline{\hat{\boldsymbol{\theta}}}}{d\theta}\frac{d\theta}{dt} \\ &= \left\{-\cos\theta\underline{\mathbf{i}} - \sin\theta\underline{\mathbf{j}}\right\}\dot{\theta} \\ &= -\left\{\cos\theta\underline{\mathbf{i}} + \sin\theta\underline{\mathbf{j}}\right\}\dot{\theta} \\ &= -\dot{\theta}\underline{\hat{\mathbf{r}}}\end{aligned}$$

Velocity in polar co-ordinates

The position of a particle at a point P in polar co-ordinates is

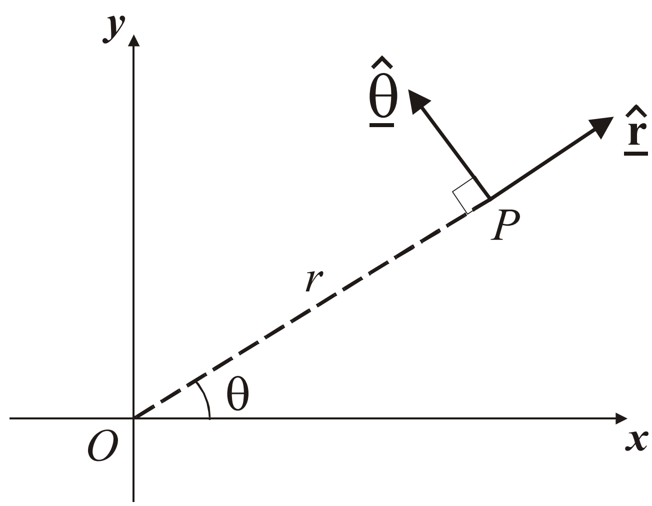
$$\underline{\mathbf{r}} = r\underline{\hat{\mathbf{r}}}$$

where r is the magnitude of the distance of P from the origin. The velocity is

$$
\begin{aligned}
\underline{\mathbf{v}} &= \frac{d\underline{\mathbf{r}}}{dt} \\
&= \frac{d}{dt} r\hat{\underline{\mathbf{r}}} \\
&= \frac{dr}{dt}\hat{\underline{\mathbf{r}}} + r\frac{d\hat{\mathbf{r}}}{dt} && \text{By the chain rule} \\
&= \dot{r}\ \hat{\underline{\mathbf{r}}} + r\dot{\theta}\hat{\underline{\boldsymbol{\theta}}} && \text{Since } \frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\underline{\boldsymbol{\theta}}}
\end{aligned}
$$

So the radical component of the velocity is $\dot{r}$ and the transverse component is $r\dot{\theta}$.

$\hat{\underline{\mathbf{r}}}$ and $\hat{\underline{\boldsymbol{\theta}}}$ are perpendicular to one another (they are "orthogonal")



Hence, the speed, v, of a particle is given by

$$v = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2}$$

Example

In one of Jules Verne's books a space craft in the form of a giant cannon ball is launched from a huge cannon. At the moment of launch, the cannon ball, containing three brave men, has radical speed $2000ms^{-1}$. The period of the earth's rotation on its own axis of revolution is 24 hours and its radius is 6,380 km. Find the speed at which our three intrepid explorers are travelling at the instant at which the cannon ball leaves the cannon.

Solution

The period is

$$T = 24 \times 60 \times 60 = 86400s$$

The angular velocity is

$$w = \dot{\theta} = \frac{2\pi}{86400} = 7.272 \times 10^{-5}\, rads^{-1}$$

The transverse component of the velocity is

$$\begin{aligned} v_\theta &= r\dot{\theta} \\ &= 6.380 \times 10^{6} \times 7.272 \times 10^{-5} \\ &= 463.9 \ \ldots.\ ms^{-1} \end{aligned}$$

The radial speed is $2000ms^{-1}$, hence the speed is

$$v = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2} = \sqrt{(463.9)^2 + (2000)^2} = 2053ms^{-1} \ \ (4.S.F.)$$

Acceleration in polar co-ordinates

To determine the acceleration of an object in polar co-ordinates we must differentiate its velocity vector.

$$\underline{v} = \dot{r}\underline{\hat{\mathbf{r}}} + r\dot{\theta}\underline{\hat{\boldsymbol{\theta}}}$$

then

$$\begin{aligned} \underline{\mathbf{a}} &= \frac{d\,\underline{\mathbf{v}}}{dt} = \frac{d}{dt}\left(\dot{r}\underline{\hat{\mathbf{r}}}\right) + \frac{d}{dt}\left(r\dot{\theta}\underline{\hat{\boldsymbol{\theta}}}\right) \\ &= \left(\ddot{r}\underline{\hat{\mathbf{r}}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt}\right) + \left(\dot{r}\dot{\theta}\underline{\hat{\boldsymbol{\theta}}} + r\ddot{\theta}\underline{\hat{\boldsymbol{\theta}}} + r\dot{\theta}\frac{d}{dt}\hat{\boldsymbol{\theta}}\right) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\underline{\hat{\boldsymbol{\theta}}} + \dot{r}\dot{\theta}\underline{\hat{\boldsymbol{\theta}}} + r\ddot{\theta}\underline{\hat{\boldsymbol{\theta}}} - r\dot{\theta}\,\dot{\theta}\underline{\hat{\mathbf{r}}} \\ &= \left\{\ddot{r} - r\dot{\theta}^2\right\}\underline{\hat{\mathbf{r}}} + \left\{r\ddot{\theta} + 2\dot{r}\dot{\theta}\right\}\underline{\hat{\boldsymbol{\theta}}} \end{aligned}$$

This means that the radical component of the acceleration is

$$\ddot{r} - r\dot{\theta}^2$$

and the transverse component is

$r\ddot{\theta} + 2\dot{r}\dot{\theta}$

Example

A beautiful girl is sitting on a horse on a merry-go-round. She has dropped her purse and is foolishly trying to reach down to pick it up. The merry-go-round has a period of 10s. In order to prevent the girl from harming herself, Larry, the experienced operator is walking towards her along a radius of the merry-go-round. The radius of the merry-go-round is 8m. At a given time Larry is 4m from the centre and is travelling $1.5ms^{-1}$. If Larry has a mass of 75 kg find the force he has to exert in a radical direction in order to maintain his speed towards the damsel in distress. Find also the force he has to exert in a transverse direction in order to maintain his balance. Find the total force he has to exert in order to maintain his course of action.

Solution

$T = 10$

Therefore $\omega = \dot{\theta} = \frac{2\pi}{T} = \frac{2\pi}{10} = 0.2\pi rads^{-1}$

The radical component of Larry's acceleration is

$$
\begin{aligned}
a_r &= \ddot{r} - r\dot{\theta}^2 \\
&= \frac{d^2}{dt^2}(0.2\pi) - 4\times(0.2\pi)^2 \\
&= 0 - 4\times(0.2\pi)^2 \\
&= 1.579...ms^{-2}
\end{aligned}
$$

Since $F = ma$

The force Larry must exert in order to maintain his constant velocity towards the girl is

$F_r = 75\times 1.579.... = 120N \ (2.S.F.)$

The transverse component of Larry's acceleration is

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

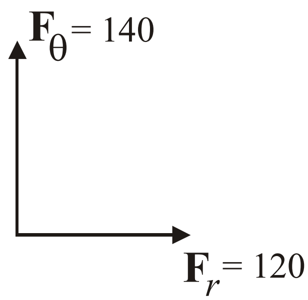
Here $r = 4 \quad \theta = 0.2\pi, \quad r = 1.5 \quad \theta = 0$

Hence

$a_\theta = 4\times 0 + 2\times 1.5\times 0.2\pi = 1.884..ms^{-2}$

Hence the force he must exert in the transverse direction is

$F_\theta = ma = 75 \times 1.884.. = 140N$



The total force is:

$$\sqrt{(F_\theta)^2 + (F_R)^2} = \sqrt{(1.18)^2 + (141)^2} = 180N \ (2.S.F.)$$

<u>Example</u>
A particle is moving with constant angular velocity $6rads^{-1}$ along a curve with equation

$r = 4\cos\theta$

Find the radical and transverse components of the acceleration in terms of θ. Find the acceleration of the particle.

Solution

We have

$\underline{\mathbf{r}} = 4\cos\theta \underline{\hat{\mathbf{r}}}$

and $\theta = 6rads^{-1}$

where

$r = 4\cos\theta$

Now

$$\dot{r} = -(4\sin\theta)\dot{\theta}$$
$$\ddot{r} = -4\cos\theta(\dot{\theta})^2 - 4\sin\theta\ddot{\theta}$$
$$\dot{\theta} = 6 \text{ and } \ddot{\theta} = 0$$

The radical component of the acceleration is

$$
\begin{aligned}
a_r &= \ddot{r} - r\theta^2 \\
&= -4\cos\theta\left(\dot{\theta}\right)^2 - \left(4\sin\theta\right)\ddot{\theta} - r\dot{\theta}^2 \\
&= \left(-4\cos\theta\right)6^2 - \left(4\sin\theta\right)\times 0 - \left(4\cos\theta\right)6^2 \\
&= -288\cos\theta
\end{aligned}
$$

The transverse component of the acceleration is

$$
\begin{aligned}
a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \\
&= r\ddot{\theta} - 2\left\{-4\sin\theta\right\}\dot{\theta}^2 \\
&= \left(4\cos\theta\right)\times 0 + 2\times\left(-4\sin\theta\right)\times 6^2 \\
&= -288\sin\theta
\end{aligned}
$$

The magnitude of the total acceleration is

$$
\begin{aligned}
a &= \sqrt{\left(a_r\right)^2 + \left(a_\theta\right)^2} \\
&= \sqrt{\left(-288\cos\theta\right)^2 + \left(-288\sin\theta\right)^2} \\
&= 288ms^{-2}
\end{aligned}
$$

So the acceleration is constant.