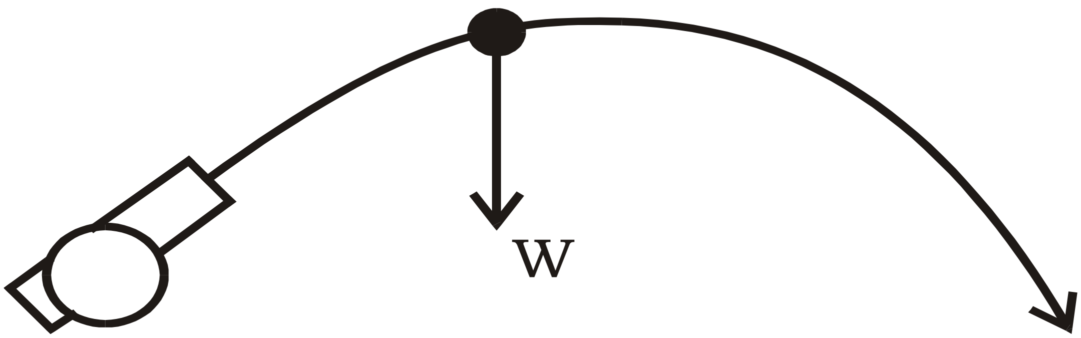
Motion of a Particle – Vector calculus form

A projectile is a particle travelling through space that has been launched in some way. A cannon ball in flight is an example of projectile. Initially, we only consider the motion of the projectile in two dimensions – horizontally and vertically. We also ignore the effect of air-resistance:



The motion of a projectile in two dimensions is resolved into horizontal and vertical components.

Since we are ignoring air-resistance, there is only one force acting on the particle, once it has been launched. This is gravity, or weight. Horizontally, it is moving with constant speed. To solve problems with projectiles we either use the equations of uniform acceleration, or we use vectors and calculus.

The use of the equations of uniform acceleration are dealt with in a separate section. Here we deal with the more powerful technique of vector calculus.

This technique is more powerful, because the equations of uniform acceleration apply only to cases where the acceleration is uniform! However, in more advanced problems we would want to consider, for example, the effect of air-resistance, which would make the force acting on the projectile variable – that is, not constant. Thus, for a student wishing to study mechanics beyond a certain level, using the vector calculus will be essential.

This is still at an introductory level, and the "full version" of the vector calculus is left to a later unit.

Vector calculus

Vectors

Displacement, velocity, acceleration and forces are vectors, and can be resolved into components. Here we take **i** as a unit vector in the horizontal direction, and **j** as a unit vector in the vertical direction.

In hand written work it is usual to underline a vector, i, and in typed text it is usual to place a vector in bold, **i.** In order to help the student learn *both* conventions we do both – so we represent a vector as both underlined and in bold.

Calculus

Acceleration is the derivative of velocity.
Velocity is the derivative of displacement.

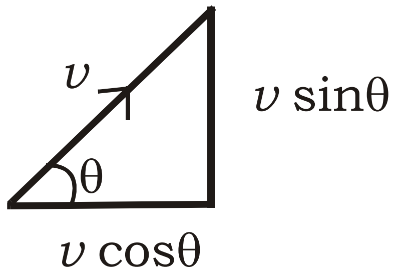
Displacement is the integral of velocity.
Velocity is the integral of acceleration.

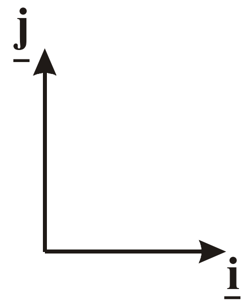
Armed with these tools we can solve problems in projectiles. However, we need to be reminded of the physical principles that also underlie problem solving in this area.

For vertical motion under gravity

A particle travels upwards because it is given, at the instant that it begins its flight, an initial velocity. For the cannon ball, this arises from the explosion in the barrel. For a golf ball its upward motion would arise from the impulse given to it by being struck by the golf-club. At no point after the projectile has been launched is it "pushed" or "pulled" by any other force than gravity. In these cases the projectile is not a rocket that carries with it its own means of propulsion. At all times it is accelerating towards the earth and it only continues to travel upwards until the gravitational force has brought its initial velocity to zero.

Using vector notation the initial velocity is resolved into horizontal and vertical components:





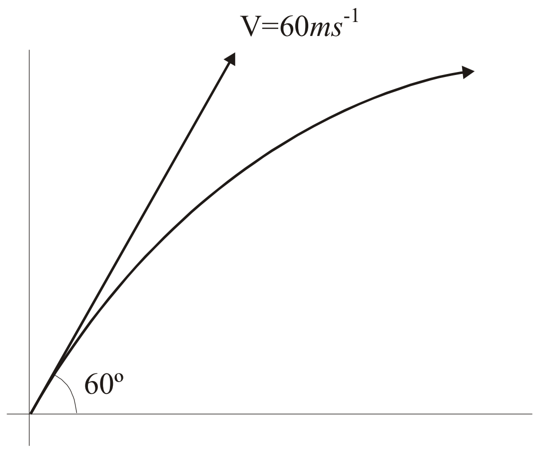
$$\underline{\mathbf{v}} = v\cos\theta\,\underline{\mathbf{i}} + v\sin\theta\,\underline{\mathbf{j}} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$$

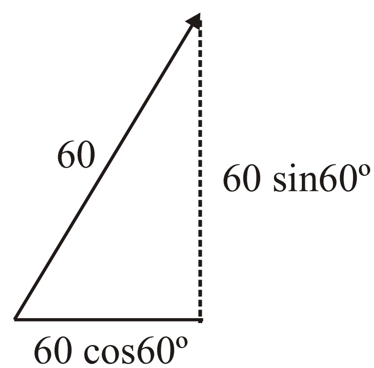
Maximum height is reached when vertical velocity = 0.

The total time of flight is twice the time taken to reach the maximum height.

Example (1)

A particle is projected with a velocity of $60ms^{-1}$ to the horizontal, and at an angle of 60° Find its velocity and the angle its velocity makes with the horizontal after $2s$.





The initial velocity (i.e. at time $t = 0$) is given by

$\underline{\mathbf{v}}_0 = 60\cos 60\,\underline{\mathbf{i}} + 60\sin 60\,\underline{\mathbf{j}}$

The acceleration under gravity is

$\underline{\mathbf{a}} = -g\underline{\mathbf{j}}$

The negative sign indicates that the object is falling under gravity, whereas the positive sign of the vertical component of the intial velocity indicates that intially it is moving upwards.

Integrating to find $\underline{\mathbf{v}}$:

$\underline{\mathbf{v}} = \int \underline{\mathbf{a}}\, dt = \int -g\, \underline{\mathbf{j}}\, dt = -\, gt\, \underline{\mathbf{j}} + c$

Using the boundary condition $\underline{\mathbf{v}}_0 = 60\cos 60\, \underline{\mathbf{i}} + 60\sin 60\, \underline{\mathbf{j}}$ to find c:

$c = 50\cos 25\, \underline{\mathbf{i}} + 50\sin 25\, \underline{\mathbf{j}}$

Therefore

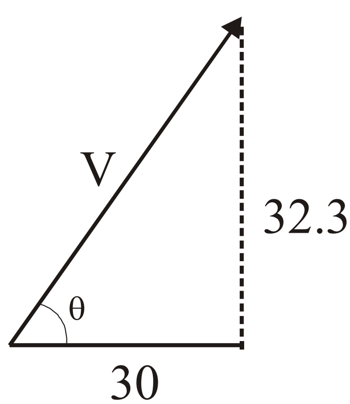
$\underline{\mathbf{v}} = 60\cos 60\, \underline{\mathbf{i}} + \left(60\sin 60 - gt\right) \underline{\mathbf{j}}$

Substituting $t = 2,\ g = 9.81$

$\underline{\mathbf{v}}(2) = 60\cos 60\, \underline{\mathbf{i}} + \left(60\sin 60 - 2\times 9.81\right) \underline{\mathbf{j}}$

$\underline{\mathbf{v}}(2) = 30\, \underline{\mathbf{i}} + (32.34...)\, \underline{\mathbf{j}}$

This describes completely the velocity of the particle at $t = 2$. The symbol $\underline{\mathbf{v}}(2)$ means the velocity at $t = 2$. We can use this equation, however, to find the magnitude of the velocity (that is, its speed) and direction (the angle it makes with the horizontal).



The magnitude of the velocity is given by

$$|\underline{\mathbf{v}}(2)| = \sqrt{32.34..^2 + 30^2} = 44.1ms^{-1} \ (3.S.F.)$$

The direction of the velocity is given by

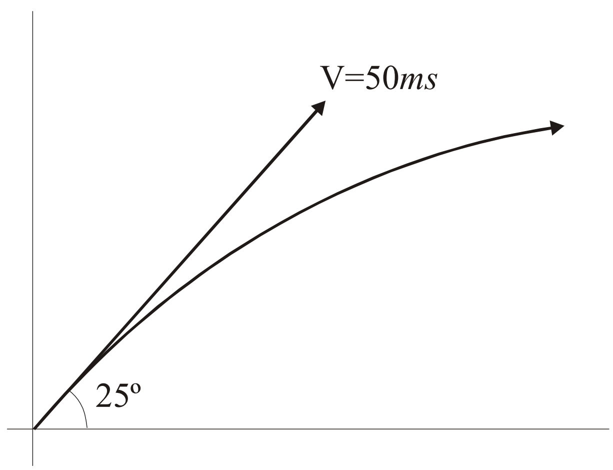
$$\tan\theta = \frac{32.4..}{30}$$

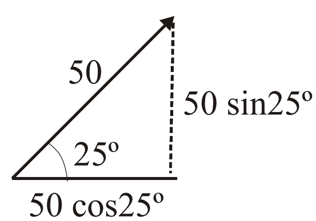
$$\theta = \tan^{-1}\left(\frac{32.4..}{30}\right) = 44.1 \ (\text{nearest } 0.1°)$$

<u>Example (2)</u>

A particle is projected with a velocity of $50ms^{-1}$ at an angle of 25° to the horizontal. What is the greatest height of the projectile above the point of projection?

Answer





When the object reaches its greatest height the vertical component of its velocity is 0. We use this fact to solve the problem, using vector calculus.
The initial velocity is given by

$$\underline{\mathbf{v}}_0 = 50\cos 25\,\underline{\mathbf{i}} + 50\sin 25\,\underline{\mathbf{j}}$$

The acceleration under gravity is

$\underline{\mathbf{a}} = -g\,\underline{\mathbf{j}}$

Integrating and applying the boundary condition as in Example 1 gives:

$\underline{\mathbf{v}} = 50\cos 25\,\underline{\mathbf{i}} + \left(50\sin 25 - gt\right)\underline{\mathbf{j}}$

Applying the condition for maximum height that the $\underline{\mathbf{j}}$ component of velocity $= 0$

$50\sin 25 - gt = 0$

$$t = \frac{50\sin 25}{g} = 2.15$$

Integrating to find $\underline{\mathbf{s}}$

$$\underline{\mathbf{s}} = \int \underline{\mathbf{v}}\,dt = \int 50\cos 25\,\underline{\mathbf{i}} + \left(50\sin 25 - gt\right)\underline{\mathbf{j}}\,dt = \left(50\cos 25\right)t\,\underline{\mathbf{i}} + \left(\left(50\sin 25\right)t - \frac{1}{2}gt^2\right)\underline{\mathbf{j}} + c$$

The boundary condition of $\underline{\mathbf{s}} = 0$ at $t = 0$ also gives $c = 0$

$$\underline{\mathbf{s}} = \left(50\cos 25\right)t\,\underline{\mathbf{i}} + \left(\left(50\sin 25\right)t - \frac{1}{2}gt^2\right)\underline{\mathbf{j}}$$

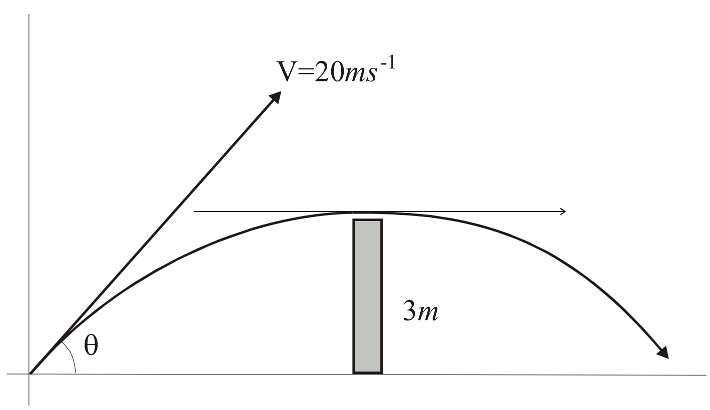
Substituting $t = 2.15$ into the $\underline{\mathbf{j}}$ component of $\underline{\mathbf{s}}$, and using $g = 9.81$

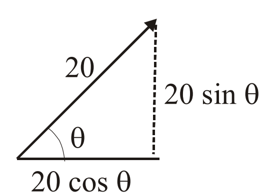
Maximum height $= 50\sin 25 \times 2.15 - \frac{1}{2} \times 9.81 \times 2.15^2 = 22.8\text{m}$

Example (3)

A golf ball is struck so that it just passes over a wall $3m$ high, at which time it is moving horizontally. The initial speed of projection is $20ms^{-1}$. What is the angle of projection?

Answer





To solve this question we must first find the time at which the particle passes over the wall. This is found using the fact that the velocity is horizontal at this point.

We start with the expression for acceleration:

$\underline{\mathbf{a}} = -g\,\underline{\mathbf{j}}$

Integrating and applying the boundary condition (as in Example 1):

$\underline{\mathbf{v}} = 20\cos\theta\,\underline{\mathbf{i}} + (20\sin\theta - gt)\,\underline{\mathbf{j}}$

Equating the $\underline{\mathbf{j}}$ component to 0 :

$20\sin\theta - gt = 0$

$t = \dfrac{20\sin\theta}{g}$

We cannot solve this equation directly, as it has two unknowns, t and θ.

We must use the fact that we know the vertical position of the particle at this point.

Integrating $\underline{\mathbf{v}}$ and applying the boundary condition (as in Example 2)

$\underline{\mathbf{s}} = (20\cos\theta)t\,\underline{\mathbf{i}} + \left((20\sin\theta)t - \frac{1}{2}gt^2\right)\underline{\mathbf{j}}$

We must equate the $\underline{\mathbf{j}}$ component of $\underline{\mathbf{s}}$ with the height of the wall, 3m.

$(20\sin\theta)t - \frac{1}{2}gt^2 = 3$

Substituting in $t = \dfrac{20\sin\theta}{g}$

$20\sin\theta\left(\frac{20\sin\theta}{9.81}\right) - \frac{1}{2}\times 9.81\times\left(\frac{20\sin\theta}{9.81}\right)^2 = 3$

$\dfrac{400\sin^2\theta}{9.81} - \dfrac{400\sin^2\theta}{2\times 9.81} = 3$

$\dfrac{400\sin^2\theta}{2\times 9.81} = 3$

$\sin^2\theta = \dfrac{3\times 2\times 9.81}{400}$

$\theta = 22.6^\circ$ (nearest 0.1°)