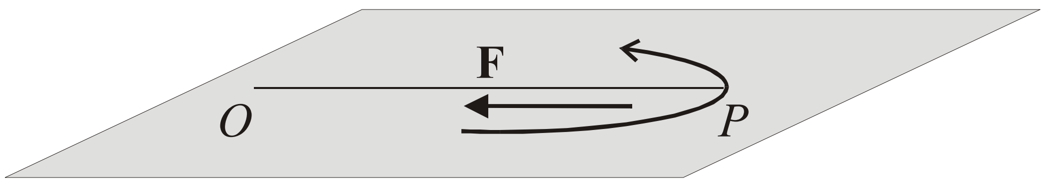
Motion under a central force

An object performs motion under a central force when the only significant force acting on that object is a force directed radically towards or away from a fixed object.

Since the only force acting on the object is the central force, the object must more in a plane.



There is no other force to lift it out of the plane.

We will also show that for an object subject to only a central force:
$r^2\dot{\theta} = h =$ constant.

Now the transverse acceleration is

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

But

$$\begin{aligned} \frac{1}{r}\frac{d}{dt}\left(r^2\theta\right) &= \frac{1}{r}\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) \\ &= 2\dot{r}\theta + r\ddot{\theta} \\ &= a_\theta \end{aligned}$$

So the transverse acceleration can also be written

$$a_\theta = \frac{1}{r}\frac{d}{dt}\left\{r^2\dot{\theta}\right\}$$

Since the transverse acceleration for an object under only a central force is 0

$$\frac{1}{r}\frac{d}{dt}\left\{r^2\dot{\theta}\right\} = 0$$

Therefore: $$\frac{d}{dt}\left\{r^2\dot{\theta}\right\} = 0$$

Hence, by direct integration

$$r^2\dot{\theta} = h$$

Where h is a constant.

Transverse Velocity

The transverse velocity of an object is

$$v_\theta = r\dot{\theta}$$

We have just seen that

$$r^2\dot{\theta} = \text{constant}$$

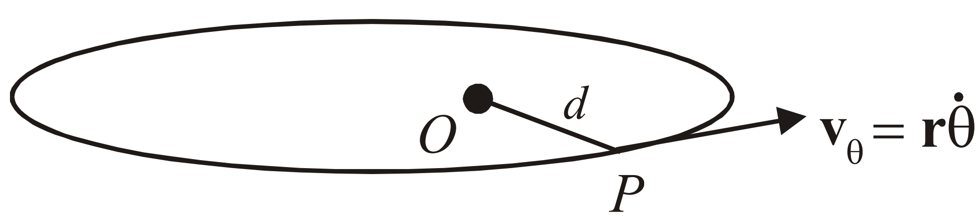
Hence

$$r\left\{r\dot{\theta}\right\} = \text{constant}$$

That is

$$rv_\theta = \text{constant}$$

This means that the distance of the object from the centre of orbit times the transverse velocity is always constant. Objects in orbit move on ellipses, where the centre of gravity lies at one focus.



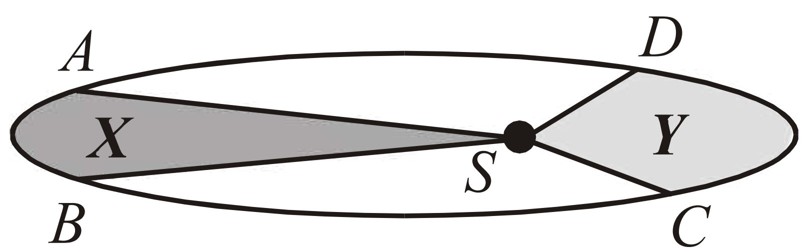
If d represents the distance of closest approach of the object to the centre of orbit, and v is the speed of the object at this point, then

$v_\theta r = vd = \text{constant} = h$

If the object has mass m then the quantity $mr^2\dot{\theta}$ represents its angular momentum. The fact that $r^2\dot{\theta}$ is constant thus indicates that for an object subject solely to a central force the angular momentum is constant.

Kepler's Second Law

Kepler's second law states that the line joining an object in orbit to the sun sweeps out equal areas in equal times.

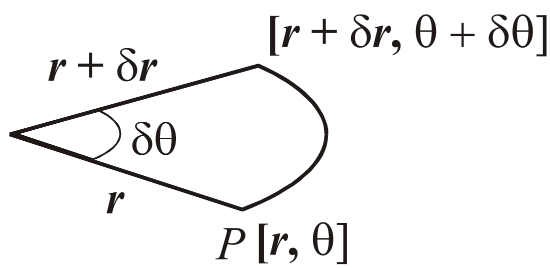


If the area X is equal to the area Y then the object in orbit takes the same time to travel from A to B as it does to travel from C to D.

Kepler's second law can be derived from the fact that for a body subject only to a central force

$r^2\dot{\theta} = \text{constant}$

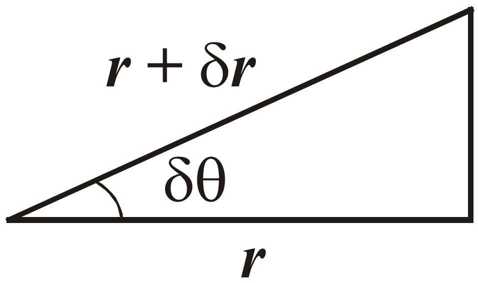
Consider a sector of an ellipse



In time δt the object P travels from

$[r,\theta]$ to $[r+\delta r, \theta + \delta\theta]$

The area of the sector swept out in this time can be approximated by a triangle



The area of the triangle is

$$\delta A \approx \frac{1}{2} r\left(r+\delta\right)\sin\left(\delta\theta\right)$$

This is an application of the usual result for a triangle with sides *a*,*b* and included angle θ that its area is $A = \frac{1}{2} ab\sin\theta$

The rate of change of the area is $\frac{\delta A}{\delta t} \approx \frac{1}{\delta t}\frac{1}{2} r\left(r+\delta r\right)\sin\left(\delta\theta\right)$

As $\delta\theta \to 0$ the approximation becomes exact; furthermore,

$$\begin{aligned} & r+\delta r \to r \\ & \sin\left(\delta\theta\right) \to \delta\theta \end{aligned}$$

So the rate of change of the area is

$$\begin{aligned} \frac{\delta A}{\delta t} &= \lim_{\delta\theta\to 0}\left\{\frac{1}{2} r\{r+\delta r\}\sin\delta\theta\right\} \\ &= \frac{1}{2} r^2 \lim_{\delta\theta\to 0}\left\{\frac{\delta\theta}{\delta t}\right\} \\ &= \frac{1}{2} r^2 \frac{d\theta}{dt} \\ &= \frac{1}{2} r^2 \dot{\theta} \end{aligned}$$

Since $r^2\dot{\theta} = \text{Constant}$, the rate of change is constant. This means that the area swept out by a radical line in any constant period of time must be constant.

Motion under gravity

The gravitational force is

$$F = \frac{GmM}{r^2}$$

It is an example of a force that obeys an inverse square law - that is, the magnitude of the force is inversely proportional to the square of the distance between the two masses. When considering a planetary system the mass of the planet can be considered to be negligible in comparison to the mass of the sun. Consequently, the planet can be regarded as moving under a central force given by

$$F = \frac{GMm}{r^2}$$

This force can be written in vector form as

$$\underline{\mathbf{F}} = -\frac{GMm}{r^2}\underline{\hat{\mathbf{r}}}$$

The negative sign indicates that the force is directed towards the centre of attraction.

We remind you that the general equation for a conic section is

$$r = \frac{1}{1 + e\cos\theta}$$

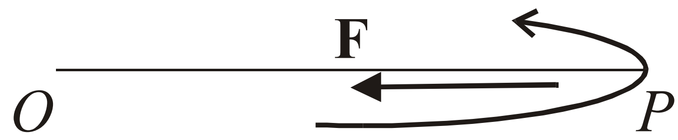
where e is the eccentricity. The magnitude of the eccentricity determines the shape of the section. Thus

$e = 0$	circle
$e < 1$	ellipse
$e = 1$	parabola
$e > 1$	hyperbola

Kepler's first law states that the path of each planet is an ellipse with the sun at one focus.

We will now derive this law - or rather, a generalisation of it, that the orbit of any object subject to only a central force that obeys the inverse square law and is directed towards the origin is an ellipse.

<u>Proof</u>



Let O be the origin and P be the object in orbit. Let $\underline{\mathbf{r}} = r\hat{\underline{\mathbf{r}}}$ represent the position of the object at time t. The object is subject to a central force that obeys the inverse square law.

$$\underline{F} = -\frac{km}{r^2}\hat{\underline{\mathbf{r}}}$$

Where k is a constant and m is the mass of the object. Newton's second law is $F = ma$. Let $\underline{\mathbf{a}}_r$ represent the radial component of the acceleration.

Therefore

$$m\underline{\mathbf{a}}_r = \frac{-km}{r^2}\hat{\underline{\mathbf{r}}}$$

Hence $\underline{\mathbf{a}}_r = \dfrac{-k}{r^2}\hat{\underline{\mathbf{r}}}$

The radial component is $\underline{\mathbf{a}}_r = \left\{\ddot{r} - r\dot{\theta}^2\right\}\hat{\underline{\mathbf{r}}}$

Hence $\ddot{r} - r\dot{\theta}^2 = \dfrac{-k}{r^2}$

It is difficult to solve this differential equation directly. Experience has shown that substituting $r = \dfrac{1}{u}$ effectively brings it into a form that can be solved. Before we do so , we use the fact that

$$r^2\dot{\theta} = h = \text{constant}$$

To derive formulae for $\dot{r}$ and $\ddot{r}$ in terms of u. Since

$$r = \frac{1}{u} = u^{-1}$$
$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

That is $\dot{r} = -\frac{1}{u^2}\dot{u}$

However $\dot{u} = \frac{du}{dt}$ can also be expressed by the chain rule as

$$\dot{u} = \frac{du}{dt} = \frac{du}{d\theta}\cdot\frac{d\theta}{dt} = \frac{du}{d\theta}\dot{\theta}$$

So $\dot{r} = \frac{-1}{u^2}\frac{du}{d\theta}\dot{\theta}$

But $r^2\dot{\theta} = h$

Therefore $\frac{\dot{\theta}}{u^2} = h$

Therefore $\dot{r} = -h\frac{du}{d\theta}$

On differentiating again

$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}\dot{\theta}$$

Once again $\dot{\theta} = u^2h^2$ so

$$\ddot{r} = -u^2h^2\frac{d^2u}{d\theta^2}$$

We substitute $r = \frac{1}{u}, \dot{\theta} = u^2h, \ddot{r} = -u^2h^2\frac{d^2u}{d\theta^2}$

Into $\ddot{r} - r\dot{\theta}^2 = \frac{-k}{r^2}$

To obtain

$$-u^2h^2\frac{d^2u}{d\theta^2} - \frac{1}{u}\left\{u^2h\right\}^2 = -ku^2$$

On simplifying and dividing by u^2

$$-h\frac{d^2u}{d\theta^2} - h^2u = -k$$

Therefore $\dfrac{d^2u}{d\theta^2} + u = \dfrac{k}{h^2}$

This is a second order, linear constant coefficient in-homogeneous differential equation. The homogeneous equation is

$$\frac{d^2u}{d\theta^2} + u = o$$

With auxiliary equation $\lambda^2 + 1 = 0$

With roots $\lambda = \pm\, i$ and solution

$$u_c = A\cos\{\theta + \varepsilon\}$$

Where A is the amplitude and ε is the phase shift. This supplies the complementary function. To find the particular function we try

$$u = m$$

Where m is a constant. Then

$$\frac{du}{d\theta} = 0 \ , \ \frac{d^2u}{d\theta^2} = 0$$

Hence on substituting into $\dfrac{d^2u}{d\theta^2} + u = \dfrac{k}{h^2}$

$$m = \frac{k}{h^2}$$

so $u_p = \dfrac{k}{h^2}$

So finally the solution is

$$u = u_c + u_p = A\{\cos\theta + \varepsilon\} + \frac{k}{h^2}$$

Where A and ε are constants determined by the initial conditions. Substituting $u = \dfrac{1}{r}$ gives

$$\frac{1}{r} = A\left\{\cos\theta + \varepsilon\right\} + \frac{k}{h^2}$$

The axes can be chosen so that $\varepsilon = 0$; hence

$$\frac{1}{r} = A\cos\theta + \frac{k}{h^2}$$

or

$$\begin{aligned} r &= \frac{1}{A\cos\theta + \dfrac{k}{h^2}} \\ &= \frac{h^2}{Ah^2\cos\theta + k} \\ &= \frac{h^2/k}{Ah^2/k\,\cos\theta + 1} \\ &= \frac{h^2}{k}\left\{\frac{1}{1 + e\cos\theta}\right\} \end{aligned}$$

where $e = \dfrac{Ah^2}{k}$

This is the polar equation of a conic. The particular form the path of the object takes {whether circle, ellipse, parabola, or hyperbola} will depend on the value of e. Determining that the phase angle $\varepsilon = 0$, requires that the x-axis acts as the major axis of symmetry of the conic. It also requires that $\theta = 0$ when $t = 0$.

This also introduces another beneficial simplification. When $t = 0, \theta = 0$ and the velocity is purely transverse, $v_0 = v(0)$.

We showed earlier that the law $r\dot{\theta}^2 = h$ could be expressed as

$v_\theta r = h$

where $v_\circ r_\circ = h$

So if we know the speed of the object of the point of closest approach - that is $v_\circ$ and $r_\circ$, then we can determine h.

Once h is known, the eccentricity of the orbit can be determined. It should be noted that the constant, k, is taken from the inverse square law, so it, too, should be given.

$$F = \frac{-k}{r^2} m\hat{\mathbf{r}}$$

This makes it possible to substitute for h and k in

$$r = \frac{h^2}{k}\left\{\frac{1}{1+e\cos\theta}\right\}$$

Also when $t = 0, r = r_\circ, \cos\theta = 1$, which gives

$$r_\circ = \frac{h^2}{k}\left\{\frac{1}{1+e}\right\}$$

Rearrangement gives

$$1+e = \frac{h^2}{kr_\circ}$$

$$e = \frac{h^2}{kr_\circ} - 1$$

Putting $h = r_\circ v_\circ$, then

$$e = \frac{r_\circ v_\circ{}^2}{k} - 1$$

Hence, when $\frac{r_\circ v_\circ{}^2}{k} < 2$ then $e < 1$

So in this case the orbit is an ellipse

If $\frac{r_\circ v_\circ{}^2}{k} = 2$ then $e = 1$

and the orbit follows a parabola.

Where $\frac{r_\circ v_\circ}{k} = 1$ then $e = 0$

and the orbit follows a circle.

Energy and central force systems

Total energy is conserved when a particle is subject only to a central force.

Gravitational potential energy is the integral of the gravitational force.

$$u = \int_{r}^{\infty} \frac{GMm}{x} dx = \left[\frac{-GMm}{r} \right]_{x}^{\infty} = \frac{-GMm}{r}$$

The kinetic energy of a particle is

$$\frac{1}{2}mv^2$$

Since total energy is conserved

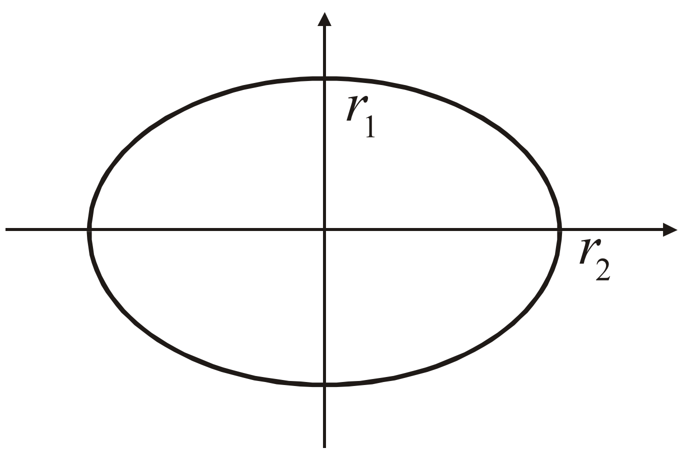
Kinetic Energy + Gravitational potential energy = Total energy

Where the total energy, E, is a constant.

Thus

$$\frac{1}{2}mr^2 \frac{-GMm}{r} = E$$

If an object under a central force is in orbit on an elliptical path, then it is possible to deduce the values r_1, and r_2 of the distance of the ellipse along the major axis



To show this, total energy is

$$E = \frac{1}{2}mv^2 - \frac{km}{r}$$

Where $F = \frac{km}{r^2}$ represents the central force.

When $r = r_1$ and $r = r_2$ the object cuts the *x*-axis, so the velocity is wholly transverse and $h = vr$, or $v = \frac{h}{r}$. Substituting into the energy equation gives

$$E = \frac{1}{2}m\left\{\frac{h}{r}\right\}^2 \frac{-km}{r}$$

Hence $2Er^2 = mh^2 - kmr$
or $2Er^2 + kmr - mh^2 = 0$

Which is a quadratic in r giving roots r_1 r_2 which are the values of r along the major axis of the elliptical orbit.