Motion under a Variable Force

Prerequisites

You should already be familiar with the formation of a differential equation, the solution to first order differential equations by the method of separation of variables and the kinematic relationships given by

$$v(t) = \frac{dx}{dt}$$
$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

where x is the displacement of a particle at time t, v is its velocity and a is its acceleration.

Example (1)

Having jumped out of an aeroplane a man is falling through the air. At time $t = 0$ his velocity is 40 ms^{-1} when he opens his parachute. His motion is then opposed by a resistive force that is a function of his velocity with magnitude $16v^2$. The man and his parachute may be modelled together as a particle of mass 80 kg.

(*a*) Show that v satisfies the differential equation

$$\frac{dv}{dt} = \frac{49 - v^2}{5}$$

(*b*) Find his terminal velocity with the parachute open.

(*c*) Solve the equation in part (*a*) and hence find the time for which the man is falling between opening his parachute and reaching a velocity of 7.25 ms^{-1}.

Solution

(*a*) The two forces acting on the man and parachute are their combined weight (W) and the resistive force $\left(R = 16v^2\right)$. The resultant force is

$$\begin{aligned} F &= W - R \\ &= mg - 16v^2 \\ &= 80 \times 9.8 - 16v^2 \\ &= 784 - 16v^2 \end{aligned}$$

By Newton's second law $F = ma = m\frac{dv}{dt}$. Hence

$$80\frac{dv}{dt} = 784 - 16v^2$$
$$\frac{dv}{dt} = \frac{784 - 16v^2}{80} = \frac{49 - v^2}{5}$$

(*b*) At the terminal velocity the resultant force and acceleration are zero. Hence

$$\frac{dv}{dt} = \frac{49 - v^2}{5} = 0$$
$$49 - v^2 = 0$$
$$v^2 = 49$$
$$v = 7 \text{ ms}^{-1}$$

(*c*) $$\frac{dv}{dt} = \frac{49 - v^2}{5}$$

Separating variables

$$5\int \frac{1}{49 - v^2}\,dv = \int dt$$

Here we are required to split $\frac{1}{49 - v^2}$ using the technique of partial fractions.

$$\frac{1}{49 - v^2} = \frac{A}{7 - v} + \frac{B}{7 + v} = \frac{A(7 + v) + B(7 - v)}{49 - v^2} \qquad \Rightarrow \qquad A = B = \frac{1}{14}$$

Hence

$$\frac{5}{14}\int \frac{1}{7 - v} + \frac{1}{7 + v}\,dv = \int dt$$
$$\frac{5}{14}\left(-\ln|7 - v| + \ln|7 + v|\right) + c = t$$
$$t = \frac{5}{14}\ln\left|\frac{7 + v}{7 - v}\right| + c$$

Substituting the boundary condition $t = 0$, $v = 40$

$$c = -\frac{5}{14}\ln\left|\frac{7 + 40}{7 - 40}\right| = -\frac{5}{14}\ln\left|\frac{47}{33}\right| = -0.1263000...$$

When $v = 7.25$ we have

$$t = \frac{5}{14}\ln\left|\frac{7 + 7.25}{7 - 7.25}\right| - 0.1263000... = 1.4439... - 0.1263... = 1.3176... = 1.32 \text{ s} \quad (3 \text{ s.f.})$$

This example shows that the introduction of a resistive force into a question that is essentially about freefall does not create exceptional complications, provided that one is familiar with differential equations and their solution by means of separation of variables.

Resistive forces

The motion of a projectile through air is subject to air-resistance. Air-resistance is typical of a *variable force*. When the projectile is stationary, there is no air-resistance. As the velocity of the projectile increases, so too does the resistance. Thus air resistance varies with velocity in some way.

The resistance to a particle through air is a matter of empirical study. In other words, the only way to discover the law governing this resistance is to conduct experiments. We can see, however, that the nature of the resistance will also depend on the shape of the particle. A streamlined particle will experience a different resistance to one that is rough and has sharp edges.

Air is an example of a *medium* through which a particle can travel. Clearly, we can consider the motion of a particle through other mediums. A particle could be moving through treacle or though oil. Thus, the resistance to the particle will depend also on the nature of the medium through which the particle is moving. There are other variables that may influence the nature of the *resistive force* exerted on a particle moving through a medium. For example, an electrically charged particle may experience a different resistance to motion than one that is not.

In this chapter we are not directly concerned with the physics that may be used to explain the nature of the resistive force acting on a particle or with the way in which an experiment might be conducted and an empirical law deduced. We are concerned primarily with the mathematical treatment of resistive forces, and in this subsection we shall consider just two cases

(1) Where the resistive force is proportional to the velocity of the particle

$R \propto v$

(2) Where the resistive force is proportional to the square of the velocity of the particle

$R \propto v^2$

In fact, as example (1) shows, we have already mastered nearly all the theory that is required for the solution of this mathematical problem. Essentially, we use Newton's second law to translate the information given into a mathematical statement. The relationships

$$v(t) = \frac{dx}{dt}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

then enable us to form a differential equation. However, there is one additional element of theory that is missing, which we now proceed to describe.

In the expression $a(t) = \frac{d^2x}{dt^2}$, acceleration is a function of time. By an application of the chain rule we can replace time as the independent variable by displacement.

The chain rule

Let $u(x) = u(v) = u(v(x))$ then

$$\frac{du}{dx} = \frac{du}{dv} \times \frac{dv}{dx}.$$

When u is a function of v, and v is a function of x, then rate of change of u with respect to x is equal the rate of change of u with respect to v multiplied by the rate of change of v with respect to x.

The formula $a = v\frac{dv}{dx}$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{dv}{dx} \times \frac{dx}{dt} && \left[\text{Using the chain rule}\right] \\ &= \frac{dx}{dt} \times \frac{dv}{dx} && \left[\text{A rearrangement}\right] \\ &= v\frac{dv}{dx} && \left[\text{Definition of velocity, } v = \frac{dx}{dt}\right] \end{aligned}$$

The formula $a = v\frac{dv}{dx}$ can be used to replace a by $v\frac{dv}{dx}$ and so solve a problem in terms of displacement rather than time. Only a worked example can make this clear.

Example (1) continued

(*d*) In part (*a*) we saw that the equation governing the motion of the man with his parachute was

$$\frac{dv}{dt} = \frac{49 - v^2}{5}$$

Use the relationship $a = v\frac{dv}{dx}$ to form a differential equation in terms of displacement *x*.

(*e*) Solve this equation to find the distance the man falls during the time it takes for his velocity to decrease from 40 ms^{-1} to 7.25 ms^{-1}.

Solution

(*d*) $$\frac{dv}{dt} = \frac{49 - v^2}{5}$$

That is

$$a = \frac{49 - v^2}{5}$$

Substituting $a = v\frac{dv}{dx}$ we obtain

$$v\frac{dv}{dx} = \frac{49 - v^2}{5}$$

(*e*) By separation of variables

$$5\int \frac{v}{49 - v^2}\,dv = \int dx$$
$$-\frac{5}{2}\ln\left|49 - v^2\right| + c = x$$

The boundary condition is $x = 0$ when $v = 40$. Hence

$$c = \frac{5}{2}\ln\left|49 - (40)^2\right| = 18.3666...$$

When $v = 7.25$ we have

$$\begin{aligned} x &= -\frac{5}{2}\ln\left|49 - v^2\right| + 18.3666... \\ &= -\frac{5}{2}\ln\left|49 - (7.25)^2\right| + 18.366... \\ &= 15.1905... \\ &= 15.2 \text{ m} \ \ (3 \text{ s.f.}) \end{aligned}$$

Example (2)

A body of mass 2 kg, initially at rest at a point *O*, moves in a horizontal straight line under the action of a horizontal force of constant magnitude 20 N and resistance to motion of variable magnitude *R* N. At time *t* s, the body is at a distance *x* from *O* and its velocity is $v \text{ ms}^{-1}$. In an effort to understand the resistance to the body's motion a student is comparing two possible models for the law governing *R*.

Linear model $R = 4v$

Quadratic model $R = 4v^2$

(*a*) Show that linear model leads $R = 4v$ leads to the differential equation

$$v\frac{dv}{dx} = 10 - 2v$$

and find a similar differential equation linking *x* and *v* for the quadratic model, $R = 4v^2$.

(*b*) The student found that when $v = 2.0$ he measured *x* to be $x = 0.40 \text{ m } (2 \text{ s.f.})$. Solve the two differential equations you found in part (*a*) and use your results to suggest which of the two models is better, stating your reason.

Solution

(*a*) The resultant force acting on the body is

$F = 20 - R$

Applying Newton's second law to this and substituting $R = 4v$

$2a = 20 - 4v$

$a = 10 - 2v$

Then given $a = v\dfrac{dv}{dx}$

$$v\frac{dv}{dx} = 10 - 2v$$

Substitution of $R = 4v^2$ leads to the differential equation

$$v\frac{dv}{dx} = 10 - 2v^2$$

(*b*) For the linear model

$$v\frac{dv}{dx} = 10 - 2v = 2(5 - v)$$

$$\frac{1}{2}\int \frac{v}{5-v}\,dv = \int dx$$

$$\frac{1}{2}\int -1 + \frac{5}{5-v}\,dv = \int dx$$

$$\frac{1}{2}\left(-v - 5\ln|5-v|\right) = x + c$$

$$x = 0,\ v = 0 \ \Rightarrow\ c = -\frac{5}{2}\ln 5$$

Hence

$$x = \frac{5}{2}\ln 5 - \frac{1}{2}\left(v + 5\ln|5-v|\right)$$

For the quadratic model

$$v\frac{dv}{dx} = 10 - 2v^2 = 2\left(5 - v^2\right)$$

$$\frac{1}{2}\int \frac{v}{5-v^2}\,dv = \int dx$$

$$-\frac{1}{4}\ln\left|5 - v^2\right| = x + c$$

$$x = 0,\ v = 0 \ \Rightarrow\ c = -\frac{1}{4}\ln 5$$

Hence

$$x = \frac{1}{4}\ln 5 - \frac{1}{4}\ln\left|5 - v^2\right|$$

When $v = 2$ the linear model predicts that

$$x = \frac{5}{2}\ln 5 - \frac{1}{2}\left(2 + 5\ln|5-2|\right) = 0.277 \text{ m } (3 \text{ s.f.})$$

The quadratic model predicts

$$x = \frac{1}{4}\ln 5 - \frac{1}{4}\ln|5-4| = \frac{1}{4}\ln 5 = 0.402 \text{ m} \quad (3 \text{ s.f.})$$

Hence, the student's result is consistent with the quadratic model but not the linear model. For this reason the quadratic model is better.

Example (3)

In this question take $g = 10\ \text{ms}^{-2}$. A projectile is shot vertically upwards through a viscous liquid that exerts a resistive force that is proportional to square of the velocity of the projectile and is given by

$$R = 0.02v^2$$

where v is the speed of the particle. Given that the mass of the projectile is 0.1 kg and its initial launch velocity is $50\ \text{ms}^{-1}$, find

(*a*) the greatest height it reaches,

(*b*) the time taken to reach this highest point.

Solution

(*a*) The projectile is moving under gravity and is also opposed by a resistive force. The resultant force, *F*, is given by

$$\begin{aligned} F &= R + W \\ &= 0.02v^2 + mg \\ &= 0.02v^2 + 0.1 \times 10 \\ &= 0.02v^2 + 1 \end{aligned}$$

Applying Newton's second law

$$F = ma$$

$$0.02v^2 + 1 = 0.1a$$

$$5a = -\left(v^2 + 50\right)$$

Using the relationship $a = v\dfrac{dv}{dx}$ where x is displacement we obtain

$$5v\frac{dv}{dx} = -\left(v^2 + 50\right)$$

We solve this differential equation by means of the separation of variables

$$\int \frac{5v}{v^2 + 50}\,dv = -\int dx$$

$$x = -\frac{5}{2}\ln\left(v^2 + 50\right) + c \qquad c\text{, constant}$$

To find c we substitute the boundary conditions $x = 0$ and $v = 50$. Hence

$$c = \frac{5}{2}\ln(2550)$$

Thus the equation of motion is

$$x = \frac{5}{2}\ln(2550) - \frac{5}{2}\ln\left(v^2 + 50\right)$$

The projectile will reach its greatest height when $v = 0$.

$$x = \frac{5}{2}\ln(2550) - \frac{5}{2}\ln(50) = 9.83 \text{ m} \quad (3 \text{ s.f.})$$

(*b*) We saw above that

$$5a = -\left(v^2 + 50\right)$$

Substituting $a = \frac{dv}{dt}$ we obtain the differential equation

$$5\frac{dv}{dt} = -\left(v^2 + 50\right)$$

$$\int \frac{5}{v^2 + 50}\, dv = -\int dt$$

The integrand on the left resembles the form $\int \frac{1}{x^2 + a^2} = \frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right) + c$. Hence

$$\frac{1}{10}\tan^{-1}\left(\frac{v}{\sqrt{50}}\right) \times \sqrt{50} = -t + c$$

Substituting the boundary conditions $t = 0$ when $v = 50$

$$c = \frac{1}{10}\tan^{-1}\left(\frac{50}{\sqrt{50}}\right) \times \sqrt{50} = \frac{\sqrt{50}}{10}\tan^{-1}\left(\sqrt{50}\right)$$

Thus

$$\frac{\sqrt{50}}{10}\tan^{-1}\left(\frac{v}{\sqrt{50}}\right) = -t + \frac{\sqrt{50}}{10}\tan^{-1}\left(\sqrt{50}\right)$$

To find the time at which the particle is at its greatest height, we substitute $v = 0$

$$t = \frac{\sqrt{50}}{10}\tan^{-1}\left(\sqrt{50}\right) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{50}\right) = 1.01137... = 1.01 \text{ s} \quad (3 \text{ s.f.})$$

Rectilinear motion

Resistive forces are examples of variable forces – they vary with the speed of a particle in motion. Here we will generalise the theory and practice we have developed in the preceding section to consider cases where

(1) Acceleration is given as a function of time, displacement or velocity
(2) Velocity is given as a function of time or displacement.

The motion under consideration shall be *rectilinear* meaning that it will be motion in a straight-line. In the following example velocity is a function of displacement.

Example (4)

A particle is moving in a straight-line. Its displacement from a fixed point O at time t s is x metres and its velocity is $v = \frac{10}{1+x}$.

Given that $x = 0$ when $t = 0$ find

(a) The value of t when $x = 2$,

(b) The acceleration when $x = 2$.

Solution

(a)

$$v = \frac{10}{1+x}$$

$$\frac{dx}{dt} = \frac{10}{1+x} \qquad \text{since velocity is } v = \frac{dx}{dt}$$

Separating variables

$$\frac{1}{10}\int (1+x)\,dx = \int dt$$

$$\frac{1}{10}\left(x + \frac{x^2}{2}\right) = t + c$$

$$t = 0,\ x = 0 \quad \Rightarrow \quad c = 0$$

$$t = \frac{1}{10}\left(x + \frac{x^2}{2}\right)$$

When $x = 2,\ t = 0.4$ s

(b)

$$v = \frac{10}{1+x}$$

Now acceleration is given by $a = v\frac{dv}{dx}$ and $\frac{dv}{dx} = -\frac{10}{(1+x)^2}$. Hence

$$a = \frac{10}{1+x} \times -\frac{10}{(1+x)^2} = -\frac{100}{(1+x)^3}$$

When $x = 2,\ a = -\frac{100}{3^3} = -\frac{100}{27} = -3.7037... = -3.70\ \text{ms}^{-2}$ (3 s.f.)

In the following example acceleration is given as a function of velocity.

Example (5)

A particle P is moving in a straight line so that its acceleration a ms^{-2} is given at time t s by $a = \frac{100}{v} - 10 \qquad v \neq 0$, where v ms^{-1} is its velocity. Given that $x = 0,\ v = 1$ when $t = 0$, find x when $v = 9$.

Solution

$$a = \frac{100}{v} - 10 = 10\left(\frac{10-v}{v}\right)$$

$$v\frac{dv}{dx} = 10\left(\frac{10-v}{v}\right) \qquad \text{since } a = v\frac{dv}{dx}$$

$$\frac{1}{10}\int \frac{v^2}{10-v}\,dv = \int dx$$

$$\frac{1}{10}\int -v - 10 - \frac{100}{v-10}\,dv = \int dx \qquad \text{by polynomial division}$$

$$-\frac{1}{10}\left(\frac{v^2}{2} + 10v + 100\ln|v-10|\right) = x + c$$

Subsituting $x = 0$ and $v = 1$

$$c = -\frac{1}{10}\left(\frac{1}{2} + 10 + 100\ln 9\right) = -23.0222...$$

$$x = -\frac{1}{10}\left(\frac{v^2}{2} + 10v + 100\ln|v-10|\right) + 23.0222...$$

When $v = 9$

$$x = -\frac{1}{10}\left(\frac{(9)^2}{2} + 90 + 100\ln|9-10|\right) + 23.0222... = 9.9722... = 9.97\text{ m} \ \ (3\text{ s.f.})$$

Acceleration as a function of displacement

When acceleration is given as a function of displacement, this leads to a number of possibilities. One case is where acceleration is proportional to displacement, but in the opposite direction to the displacement. This is a special case in itself and leads to a kind of motion that is called *simple harmonic motion*, which is a topic for a further chapter. In general where acceleration is a function of displacement the solution to the differential equations that arise are beyond the scope of the techniques employed in this chapter.


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