Mutually exclusive and independent events

Prerequisites

When we introduced probability we defined mutually exclusive and independent events. We subsequently examined the subject of conditional probability. The purpose of this chapter is to re-examine the definitions of mutually exclusive and independent events in the context of conditional probability. Before we do so, let us remind you of the following three properties of probability.

(**1**) The probability of an event occurring is a number between 0 and 1 inclusive.

$0 \le P(A) \le 1$

(**2**) Let A be an event and A' denote the event that A does not occur. Then $P(A') = 1 - P(A)$.

(**3**) Let A and B be events. Then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

You should be aware of the definition of conditional probability as $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

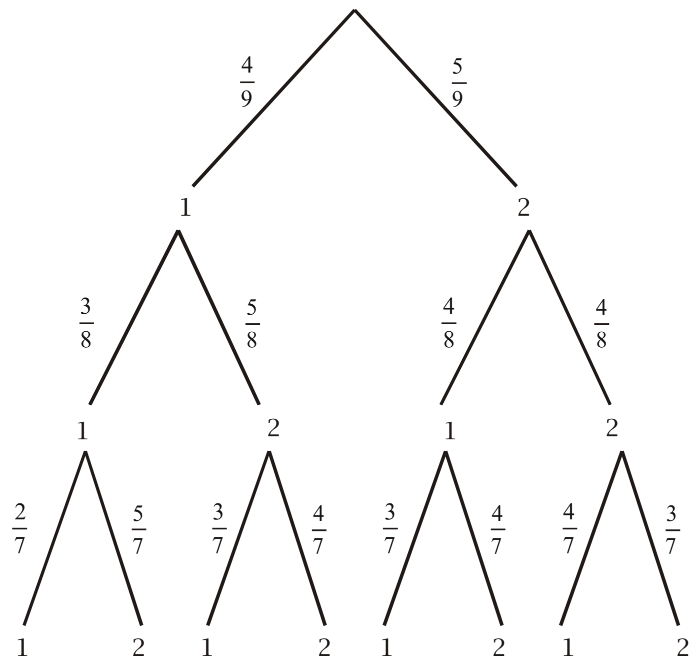
Example (1)

A bag contains 9 chips. On 4 of these chips the number 1 is written. On 5 of these chips the number 2 is written. Three chips are drawn at random from the bag without replacement. Let A denote the event that the sum of the numbers written on the chips is even. Let B denote the event that all three chips have the same number. Evaluate

(a) $P(A)$
(b) $P(A')$
(c) $P(B)$
(d) $P(A \cap B)$
(e) $P(A \cup B)$
(f) $P(A|B)$

Solution

In is not absolutely necessary to draw the entire probability tree for this problem but we shall do so nonetheless in order to illustrate the solution.



Sum	3	4	4	5	4	5	5	6
Event A		A	A		A			A
Event B	B							B

From the tree we see that

(a) $\quad P(A) = 4 \times \frac{4}{9} \times \frac{3}{8} \times \frac{5}{7} = \frac{10}{21}$

(b) $\quad P(A') = 1 - \frac{10}{21} = \frac{11}{21}$

(c) $\quad P(B) = \left(\frac{4}{9} \times \frac{3}{8} \times \frac{2}{7}\right) + \left(\frac{5}{9} \times \frac{4}{8} \times \frac{3}{7}\right) = \frac{1}{6}$

(d) $\quad P(A \cap B) = \left(\frac{5}{9} \times \frac{4}{8} \times \frac{3}{7}\right) = \frac{5}{42}$

(e) $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{10}{21} + \frac{1}{6} - \frac{5}{42} = \frac{11}{21}$

(f) $\quad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{5}{42}}{\frac{1}{6}} = \frac{5}{7}$

Defining mutually exclusive and independent events

Mutually exclusive events

Two events, A and B, are mutually exclusive if it is not possible to have both A and B. Then

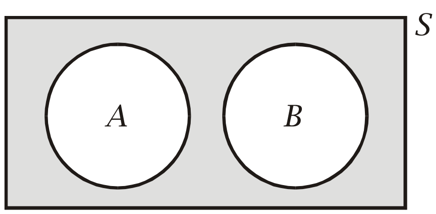
$A \cap B = \varnothing$

$P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

<u>Proof</u>

If two events A and B are mutually exclusive then as the following Venn diagram illustrates there are no outcomes to be found in both A and B.



This observation immediately entails

$A \cap B = \varnothing$

$P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

This definition of mutually exclusive events can also be expressed as

$P(A \textbf{ and } B) = 0$

$P(A \textbf{ or } B) = P(A) + P(B)$

Example (1) continued

In example (1) with events A and B defined as before

(g) Determine whether A and B are mutually exclusive.

(h) Let C denote the event that the sum of the numbers is 3. Determine

(1) Whether A and C are mutually exclusive.

(2) Whether B and C are mutually exclusive.

Solution

(g) A and B are not mutually exclusive. The outcome $(2,2,2)$ belongs to both events.

(h) A and C are mutually exclusive. The sum cannot be both even and equal to 3.

B and C are not mutually exclusive. The outcome $(1,1,1)$ belongs to both events.

Independent events

Two events A and B are independent if the knowledge that event A has occurred does not affect the probability of B occurring. The criterion for independent events is

$$P(A|B) = P(A)$$

This also entails

$$P(A \cap B) = P(A) \times P(B)$$

<u>Proof</u>

The definition of conditional probability is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (1)$$

If A and B are independent events then $P(A|B) = P(A)$.

On substitution into (1)

$$P(A) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A) \times P(B)$$

This can also be expressed in the form $P(A \textbf{ and } B) = P(A) \times P(B)$.

Example (1) continued

In example (1) with events A and B defined as before

(g) Determine whether A and B are independent events.

Solution

(g) $P(A) \times P(B) = \frac{10}{21} \times \frac{1}{6} = \frac{5}{63}$

$P(A \cap B) = \frac{5}{42}$

$P(A) \times P(B) \neq P(A \cap B)$

Not independent

Example (2)

Two fair dice are thrown. Events A, B and C are defined as follows

A the sum of the two scores is odd

B both scores are 1 or at least one score is 6

C the two scores are equal

Determine which of any two of these three events are

(a) independent and (b) mutually exclusive.

Solution

To solve this problem let us first make a diagram of the sample space as follows.

		first die					
		1	2	3	4	5	6
second die	1	CB	A		A		AB
	2	A	C	A		A	B
	3		A	C	A		AB
	4	A		A	C	A	B
	5		A		A	C	AB
	6	AB	B	AB	B	AB	CB

$P(A) = \frac{18}{36} = \frac{1}{2}$ $\quad$ $P(B) = \frac{12}{36} = \frac{1}{3}$ $\quad$ $P(C) = \frac{6}{36} = \frac{1}{6}$

$P(A \cap C) = 0$ $\quad$ $P(A \cap B) = \frac{6}{36} = \frac{1}{6}$ $\quad$ $P(B \cap C) = \frac{2}{36} = \frac{1}{18}$

<u>Mutual exclusivity</u>

$P(A \cap C) = 0 \quad \Rightarrow \quad$ A and C are mutually exclusive

$P(A \cap B) \neq 0 \quad \Rightarrow \quad$ A and B are mutually exclusive

$P(B \cap C) \neq 0 \quad \Rightarrow \quad$ B and C are mutually exclusive

<u>Independence</u>

$$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} = P(A \cap B)$$

A and B are independent

$$P(A) \times P(C) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} \neq P(A \cap C) = 0$$

A and C are not independent

$$P(B) \times P(C) = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18} \neq P(B \cap C) = \frac{2}{18}$$

Therefore, B and C are not independent

Example (3)

The events A and B are such that

$P(A) = 0.5$ $\quad$ $P(B) = 0.25$ $\quad$ $P(A|B) = 0.6$

(a) State with a reason whether or not A and B are independent.

(b) Evaluate

- (i) $P(A \cap B)$
- (ii) $P(A \cup B)$
- (iii) $P(A | A \cup B)$

Solution

(a) $P(A|B) \neq P(A)$ Not independent

(b) (i) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$0.6 = \frac{P(A \cap B)}{0.25}$$

$$P(A \cap B) = 0.15$$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.5 + 0.25 - 0.15 = 0.6$$

(ii) $P(A|A \cup B) = \dfrac{P(A \cap (A \cup B))}{P(A \cup B)}$

$$= \frac{P(A)}{P(A \cup B)}$$

$$= \frac{0.5}{0.6} = \frac{5}{6}$$

Example (4)

Show that the criterion for independent events

$$P(A|B) = P(A)$$

entails the definition

$$P(B|A) = P(B)$$

<u>Proof</u>

From the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (1)$$

Given

$$P(A \cap B) = P(A) \times P(B)$$

then, on substituting this into (1)

$$P(A|B) = \frac{P(A) \times P(B)}{P(B)} = P(A)$$

Algebra of probability

The last example illustrates how the various results regarding probability can be assembled into a single problem.

Example (5)

The events A and B are independent events such that

$P(A) = 0.6 \qquad P(A \cup B) = 0.76$

Evaluate

(a) $P(B)$

(b) $P(A \cap B')$

(c) $P(A \cap B' | B')$

Solution

(a) We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

As A and B are independent events

$$P(A \cap B) = P(A) \times P(B)$$

Hence

$$P(A \cup B) = P(A) + P(B) - P(B) \times P(A)$$

$$0.76 = 0.6 + P(B) - 0.6 \times P(B)$$

$$P(B)(1 - 0.6) = 0.16$$

$$P(B) = \frac{0.16}{0.4} = 0.4$$

(b) $P(A \cap B) = P(A) \times P(B) = 0.6 \times 0.4 = 0.24$

$$P(A \cap B') = P(A) - P(A \cap B) = 0.6 - 0.24 = 0.36$$

(a) $P(B') = 1 - P(B) = 1 - 0.4 = 0.6$

$$P(A \cap B' | B') = \frac{P(A \cap B')}{P(B')} = \frac{0.36}{0.6} = 0.6$$