The Normal Approximation to the Poisson Distribution

Prerequisites

You should be familiar with the normal distribution as an approximation to the binomial distribution and the use of the continuity correction. You should also be familiar with using linear combinations (multiples) of a Poisson distribution and with the use of the Poisson distribution to approximate a binomial distribution.

Example (1)

At a garage the probability selling a one-litre bottle of distilled water in any 1 hour is 0.4. The garage is open 40 hours a week and 8 hours a day.

(*a*) (*i*) Using a binomial distribution find the probability that the garage will sell exactly 16 one-litre bottles in one randomly chosen week.

(*ii*) Use a distributional approximation to find the probability that the garage will sell more than 90 one-litre bottles of distilled water during a randomly chosen 200 hour period of opening time.

At another garage the probability of selling a one-litre bottle in any 1 hour is 0.05. This garage is open 60 hours a week. Use a Poisson approximation to find

(*b*) (*i*) The probability that the garage will sell exactly 4 one-litre bottles in one randomly chosen week,

(*ii*) The probability the garage will sell more than 16 one-litre bottles in one randomly chosen 5-week period.

Solution

(*a*) (*i*) Let X denote the number of one litre bottles sold in one week.

$X \sim B(40, 0.4)$

$$P(X=16)=\binom{40}{16}(0.4)^{16}(0.6)^{24}=0.1279...=0.128 \text{ (3 s.f.)}$$

(*ii*) $X \sim B(200, 0.4)$ $\quad n=200 \quad p=0.4$

$n>30 \quad p\approx 0.5$

Therefore we may approximate by $X \sim N(np, npq) = N(80, 48)$

By the continuity correction $P(X > 90)$ under $X \sim B(200, 0.4)$

corresponds to $P(X > 90.5)$ under $X \sim N(80, 48)$

$x = 100.5 \qquad z = \dfrac{x-\mu}{\sigma} = \dfrac{90.5-80}{\sqrt{48}} = 1.516 \qquad \Phi(1.516) = 0.9352$

$P(X > 90) \approx P(Z > 1.516) = 1 - 0.9352 = 0.0648 = 6.48\%$ (3 s.f.)

(*c*) (*i*) Let Y denote the number of one litre bottles sold in one week.

$Y \sim B(60, 0.05) \qquad n = 60 \qquad p = 0.05$

$\mu = np = 60 \times 0.05 = 3 < 5 \qquad n > 50$

Therefore, we may approximate by $Y \sim Po(3)$

$P(Y = 4) = e^{-3}\dfrac{(3)^4}{4!} = 0.1680... = 0.168$ (3 s.f.)

(*ii*) Over 1 week we have $Y \sim Po(3)$

Therefore, over 5 weeks $5Y \sim Po(15)$

$P(Y > 16) = 1 - P(Y \le 16) = 1 - 0.6641 = 0.3359 = 0.336$ (3 s.f.)

In this question we see the utility of approximating the binomial distribution firstly by the normal distribution when finding cumulative probabilities for large n, and then by the Poisson distribution when the probability of success is small and also when n is large. In the second case the low probability of success (in the example $p = 0.05$) renders the direct use of the normal distribution unsound as an approximation. To use the normal distribution as an approximation to the binomial distribution we require that $p \approx 0.5$.

There remains one missing link in this chain of approximations. Suppose in example (1) we had gone on to ask the question about a probability over an even larger period than 5 weeks. Say, for example, we are interested in the likelihood as to whether the garage will sell more than 150 one litre bottles over a 52 week period. In this case, when we use the Poisson distribution we encounter a practical difficulty.

Example (1) continued

Using the Poisson approximation found in part (*b*) of the example (1) above investigate the probability the garage will sell more than 150 one-litre bottles in one randomly chosen 52-week period. Find the Poisson approximation suitable for this problem and explain what practical difficulty you encounter.

Solution

As before over a 1-week period $Y \sim Po(3)$. Therefore over a 52-week period $52Y \sim Po(3 \times 52) = Po(156)$. We require $P(52Y > 150)$. The practical problem is that most tables do not have entries for Poisson parameter $\lambda = 156$. We would also hardly wish to overcome this practical difficulty by brute force calculation using the definition of the Poisson distribution as $P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}$ for $x = 0,\ 1,\ 2,\ 3, ...$ by substituting $\lambda = 156$ and summing over values from 0 to 150. Therefore, to deal with this situation we are looking for another approximation. We wish to be able to replace the Poisson distribution by another distribution whenever λ is large.

The Normal approximation to the Poisson distribution

The result we are looking for is

The normal approximation to the Poisson distribution

Given

(1) $X \sim Po(\lambda)$

(2) λ is large $(\lambda > 20)$

then X may be approximated by $X \sim N(\lambda, \lambda)$

The continuity correction should be used.

Example (1) continued

Using the Poisson approximation found in part (*b*) of the example (1) find the probability the garage will sell more than 150 one-litre bottles in one randomly chosen 52-week period.

Solution

Over a 1-week period $Y \sim Po(3)$.

Over a 52-week period $52Y \sim Po(3 \times 52) = Po(156)$.

We require $P(52Y > 150)$.

Since $\lambda > 20$, we can approximate Y by $R \sim N(156, 156)$.

Using the approximation with the continuity correction

$P(52Y > 150) \approx P(R > 150.5)$

$r = 150.5 \qquad z = \frac{r-\mu}{\sigma} = \frac{150.5-156}{\sqrt{156}} = -0.440 \quad \Phi(0.440) = 0.6700$

$P(52Y > 150) \approx P(R > 150.5) = 0.670 \ \ (3 \text{ s.f.})$

Example (2)

The number of visitors to an Internet website in a day follows a Poisson distribution. On average there are 30 visitors in a given day. Use the normal approximation to the Poisson distribution to find the probability of the website receiving between 25 and 32 visitors inclusive in a given day.

Solution

$X \sim Po(30)$

Since $\lambda > 20$, we can approximate X by $Y \sim N(30,30)$

That is, $\lambda = 30$ and $\sigma = \sqrt{30}$. We require $P(25 \leq X \leq 32)$

Using the approximation with the continuity correction this is approximated by

$P(25 \leq X \leq 32) \approx P(24.5 < Y < 32.5)$

$y_1 = 24.5 \qquad z_1 = \frac{y_1-\mu}{\sigma} = \frac{24.5-30}{\sqrt{30}} = -1.0041...$

$y_2 = 32.5 \qquad z_2 = \frac{y_2-\mu}{\sigma} = \frac{32.5-30}{\sqrt{30}} = 0.4565...$

$$\begin{aligned} P(25 \leq X \leq 32) &\approx P(24.5 < Y < 32.5) \\ &= P(-1.0041... < Z < 0.4565...) \\ &= 0.3422 + 0.1761 \\ &= 0.5183 \\ &= 51.8\% \ \ (3 \text{ s.f.}) \end{aligned}$$


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