Numerical methods for solutions to differential equations

In this section we consider first order differential equations of the form

$$\frac{dy}{dx} = f(x, y)$$

where y is a function of x; that is

For example

$$\frac{dy}{dx} = xy$$

is a first order differential equation of this type. It can in fact be solved exactly by separation of variables, but exact solutions of this very general type cannot always be obtained, in which case a numerical method is required.

A numerical method will provide an approximate plot of the function $y = y(x)$ that satisfies the relationship

$$\frac{dy}{dx} = f(x, y)$$

To see what this means let us first solve and sketch the solution to our example

$$\frac{\delta y}{\delta x} = xy$$

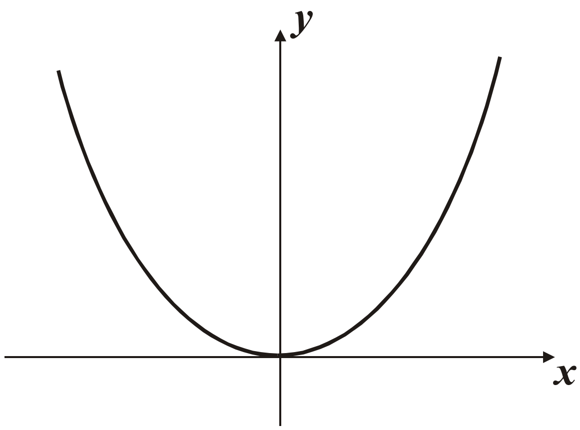
Separating variables

$$\int \frac{1}{y} dy = \int x dx$$

$$\ln y = \frac{x^2}{2} + c$$

$$y = e^{x^2/2 + c} = Ae^{x^2/2}$$

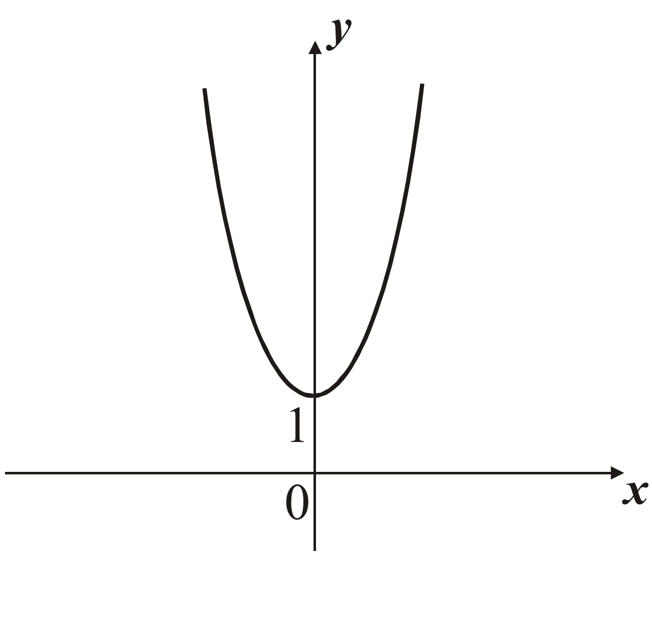
$\frac{x^2}{2}$ is a parabola with minimum 0 at $x = 0$



When $x = 0$ $y = e^{h^2/2}\Big|_{x=0} = e^0 = 1$

When
$x \to +\infty \quad x^2/2 \to +\infty \quad \therefore y = e^{x^2/2} \to \infty$
$x \to -\infty \quad x^2/2 \to +\infty \quad \therefore y = e^{x^2/2} \to \infty$

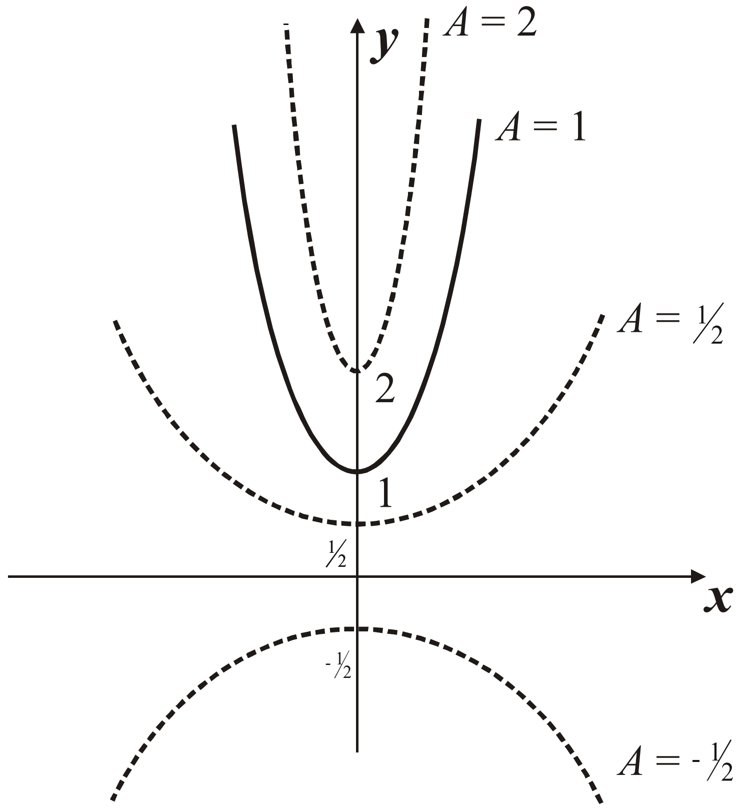
There are no turning points other than $x = 0$, so a sketch of $y = e^{x^2/2}$ is



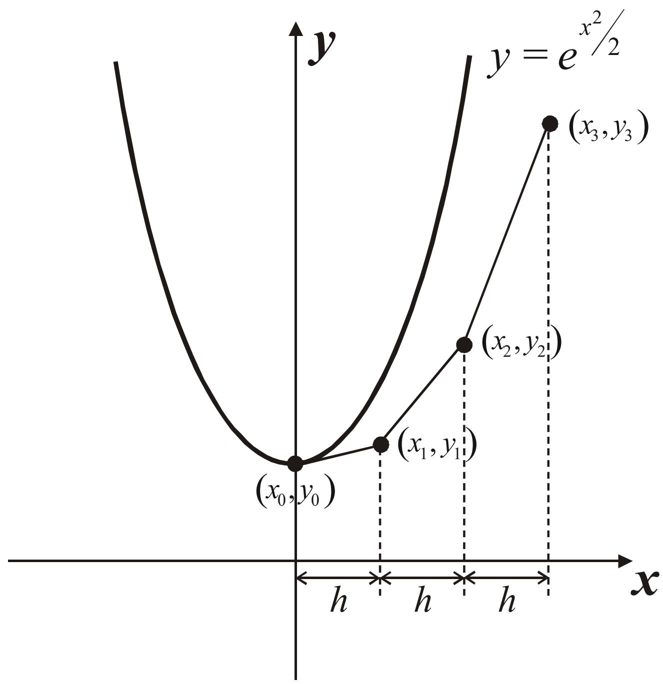
It is a curve with minimum at $x = 0$ rising very steeply on both sides of the y-axis, and also symmetrical.

The graph of $y = Ae^{x^2/2}$ is derived from the graph of $y = e^{x^2/2}$ by a scaling factor of A.

The scale factor depends on initial co-ordinates, so effectively the relationship $y = Ae^{x^2/2}$ defines a family of curves.



Now a numerical solution would be a plot of a curve that comes close to the real curve. The plot would start at some point on the curve, say (x_0, y_0) and find a sequence of subsequent points $(x_1, y_1), (x_2, y_2), \ldots\ldots., (x_i, y_i)$ lying close to the real curve. For example, an approximation to the curve $y = e^{x^2/2}$ passing through $(0,1)$ might look like



In other words a numerical solution to a differential equation is a process that starting with a point (x_0, y_0) and an interval width h, so that $x_1 = x_0 + h,$ generates a sequence of points $(x_0, y_0) \to (x_1, y_1) \to (x_2, y_2) \to ....$

And these points will lie close to the original function.

The rule for generating successive x co-ordinates is simple-just add the interval width, h, to the previous value That is $x_{n+1} = x_n + h$. What is required is a recurrence relation that generates successive y coordinates, so that y_{n+1} will be defined exclusively in terms of previous values of x and y. The first method for finding such a recurrence relation that we will consider is Euler's method.

Euler's Method

Recall that we are looking for a numerical solution to the 1st order differential equation

$$\frac{dy}{dx} = f(x, y)$$

It will be convenient to write $y' = \frac{dy}{dx}$

So our equation to be solved becomes

$$y' = f(x, y)$$

Let x_n and x_{n+1} be successive x values in our approximation that is separated by interval width h.

Then integrating both sides of this expression over that interval gives

$$\int_{x_n}^{x_{n+1}} y' dx = \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$

The left –hand side can be integrated directly so

$$\left[y\right]_{x_n}^{x_{n+1}} = \int_{x_n}^{x_{n+1}} f(x, y) dx$$

hence=

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y) dx \qquad (1)$$

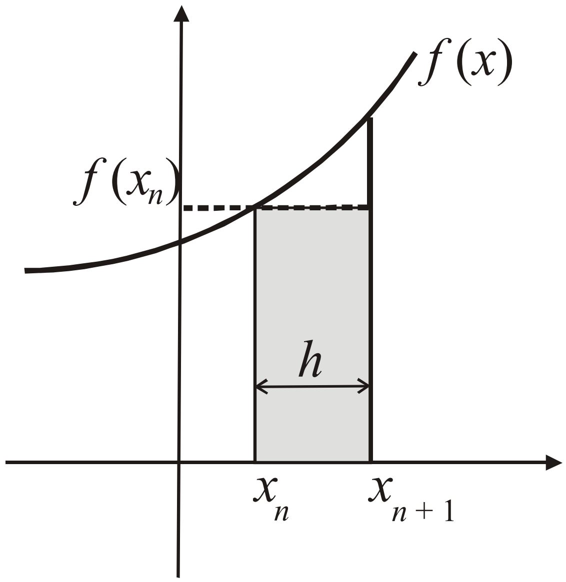
We define the y_n co-ordinates to be $y(x_n)$, so

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x, y) dx$$

we have seen that Euler's method provides a means of approximating an integral type

$$\int_{x_n}^{x_{n+1}} f(x) dx$$

where $x_{n+1} - x_n = h$, the interval width.
The approximation is by means of a rectangle.



The rectangle has area $hf(x_n)$

So $\int_{x_n}^{x_{n+1}} f(x)dx \approx hf(x_n)$

There is no reason why we should not generalize this to functions of two variable $f(x, y)$ and hence, by Euler's method

$$\int_{x_n}^{x_{n+1}} f(x)\,dx \approx hf(x_n, y_n)$$

Now we replace the right hand side of equation (1) by this approximation to obtain $y_{n+1} - y_n \approx hf(x_n, y_n)$

Rearranging, we obtain our recurrence reaction as

$$y_{n+1} = y_n + hf(x_n y_n)$$

In summary

Starting with an initial point (x_0, y_0) and an internal width h, the approximation solution by Euler's method to the differential equation.

$$\frac{dy}{dx} = f(x, y)$$

is provided by the recurrence relation

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hf(x_n, y_n)$$

Example

Find an approximate solution to $\frac{dy}{dx} = xy$

starting at $(x_0, y_0) = (0,1)$ with interval width 0.05. Plot the first four points of your solution and compare the y-value at x_3 with the exact solution and calculate the relative error in each case

Solution

Firstly, recall that the exact solution to

$$\frac{dy}{dx} = xy$$

is $y = Ae^{x^2/2}$.

The initial conditions $x_0 = 0, y_0 = 1$ *makes* $A = 1$. so $y = e^{x^2/2}$ is the exact, particular function.

Euler's method is

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hf(x_n y_0)$$

Here $h = 0.05$ so

$$x_1 = x_0 + 0.05$$
$$y_1 = y_0 + 0.05 x_0 y_0$$

$x_1 = 0 + 0.05 = 0.05$

$y_1 = 1 + 0.05 \times 0 = 1.00$

$x_2 = x_1 + 0.05$

$= 0.05 + 0.05 = 0.10$

$y_2 = y_1 + 0.05 x_1 y_1$

$= 1.00 + 0.05 \times 0.05 \times 1.00$

$= 1.0025$

$x_3 = x_2 + 0.05$

$= 0.10 + 0.05$

$= 0.15$

$y_3 = y_2 + 0.05 x_2 y_2$

$= 1.0025 + 0.05 \times 0.10 \times 1.0025$

$= 1.00500625$

Since the exact solution is $y = e^{x^2/2}$ we obtain the following exact solutions (to $8.S.F.$)

$x_0 = 0 \qquad y_0 = 1$

$x_1 = 0.05 \qquad y_1 = 1.00125078$

$x_2 = 0.10 \qquad y_2 = 1.00501252$

$x_3 = 0.15 \qquad y_3 = 1.01131352$

At $(x_0 y_0)$ the % relative error is zero

$$\% \text{ relative error } = \frac{\text{absolute error}}{\text{time value}} \times 100\%$$

at (x_1, y_1) % relative error $= \dfrac{1.00125078 - 1.00}{1.00125078} = 1.25\% \ (3.S.F.)$

at (x_2, y_2) % relative error $= \dfrac{1.00501252 - 1.00500625}{1.00501252} = 2.50\% \ (3.S.F.)$

at (x_3, y_3) % relative error $= \dfrac{1.01131352 - 1.00500625}{1.0113152} = 6.24\% \ (3.S.F.)$

So, as expected the relative error is increasing quite rapidly. We next turn our attention to the truncation error in Euler's method.

In order to do this we will introduce the symbol $f(x) = O(x)$. This is read "$f(x)$ is the order of ". What this means is that the growth of $f(x)$ is proportional to x, that is

$$f(x) = 0(k) \text{ implies } f(x) \propto x$$

This symbol (called the "big *oh* notation") is used to describe and analyze errors. The expression

$$f(x) = 0(x^2)$$

means $f(x)$ is proportional to x^2 and so forth.

In Euler's method, we had

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x, y) dx$$

We used Euler's method to approximate the right hand side, giving

$$y_{n+1} - y_n = hf(x, y) + O(h^2)$$

That is, the error terms for a single step is of the order of h^2 (that is proportional to h^2). This is the local truncation error, because it applies to a single step. The global truncation error is the error arising from n steps. It is approximately proportional to nh^2

i.e.
global truncation error $= O(nh^2)$

But the number of steps, n, is a function of the step size h. If the total interval over which the function is approximated is $x_n - x_0$ then $n = \frac{x_n - x_0}{h}$

so

global truncation error $= O\left(\frac{(x_n - x_0)h^2}{h}\right) = O\big((x_n - x_0)h\big) = O(h)$

since $x_n - x_0$ is a constant not depending on h. So the error carried over from one step to another is proportional to the step size.

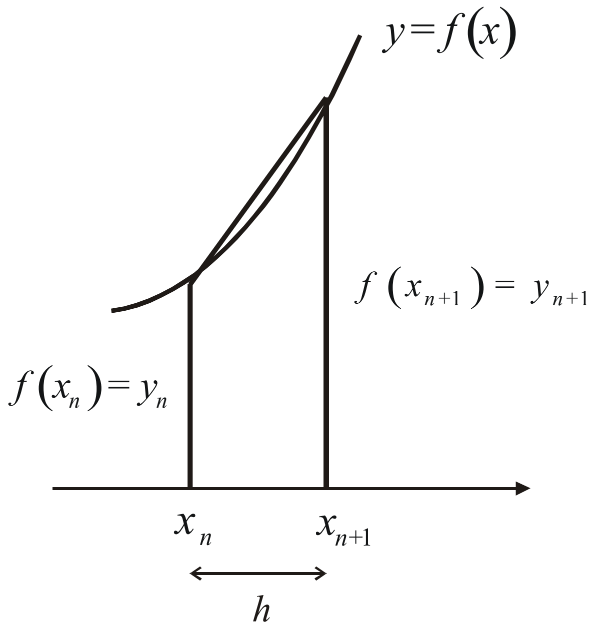
The Euler Trapezoid Method (Runge-Kutla method)

The global truncation error for Euler's method is of the order of h, which means that accuracy can only be obtained if h is small, which correspondingly means that there is a greater risk of rounding errors. So alternative methods are sought with lower truncation errors. One such method is the trapezoidal method. Recall that in a searching for a recursive process to solve.

$$\frac{dy}{dx} = f(x,y)$$

we arrived at $y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x,y)dx$

We approximated the right hand side by a rectangle, as in Euler's method for numerical integration. In the trapezoid method we use a trapezium.



Using the trapezoid method would give

$$\int_{x_n}^{x_{n+1}} f\left(x,y\right) \approx \frac{1}{2}h\left(f\left(x_n y_n\right)\right)+f\left(x_{n+1},y_{n+1}\right)$$

which on substitution into

$$y_{n+1}-y_n=\int_{x_n}^{x_{n+1}} f\left(x,y\right)dx$$

gives

$$y_{n+1}-y_n=\frac{1}{2}h\left(f\left(x_n,y_n\right)\right)+f\left(x_{n+1},y_{n+1}\right)\ \ )$$

Unfortunately , the right hand side of this equation contains a term $f\left(x_{n+1}y_{n+1}\right)$ which is dependant on the very value, y_{n+1} that we are seeking to approximate. We can get around this by approximating y_{n+1} using Euler's method

$$y_{n+1}=y_n+hf\left(x_n,y_n\right)$$

which gives

$$y_{n+1}-y_n=\frac{1}{2}h\left\{f\left(x_n,y_n\right)+f\left(x_{n+1},\left(y_n+hf\left(x_n,y_n\right)\right)\right\}\right.$$

or approximation

$$y_{n+1}=y_n+\frac{1}{2}h\left\{f\left(x_n,y_n\right)+f\left(x_{n+1},\left(y_n+hf\left(x_n,y_n\right)\right)\right\}\right.$$

The recursion process

$$x_{n+1}=x_n+h$$
$$y^*{}_{n+1}=y_n+hf\left(x_n y_n\right)$$
$$y_{n+1}=y_n+\frac{1}{2}h\left(f\left(x_n,y_n\right)+f\left(x_n,y^*{}_{n+1}\right)\right)$$

Effectively the Euler-trapezoid method finds firstly, an approximation to y_{n+1} using Euler's method and the improves upon it by means of trapezoidal method for numerical integration. For this reason the method is also called the predictor–corrector method. It is also called the Runge-Kutta method after the German mathematician Carl Runge who devised it. It was later modified by Kutta.

<u>Example</u>

Find a numerical solution to $\frac{dy}{dx} = xy$

with $(x_0 y_0) = (0,1)$ for $x_0 = 0$ $x_, = 0.05$
$x_2 = 0.10$ and $x_3 = 0.15$
by means of the Runge-Kutta method. Calculate the relative error in the y-coordinates in each case and compare with Euler's method.

Solution

Using the trapezoidal method

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hf(x_n y_n)$$
$$y_{n+1} = y_n + \frac{1}{2}h\left(f(x_n y_n) + f(x_{n+1}, y_{n+1})\right)$$

That is

$$x_{n+1} = x_n + 0.05$$
$$y_{n+1} = y_n + 0.05 x_n y_n$$
$$y_{n+1} = y_n + 0.025(x_n y_n + y_{n+1})$$

so

$$x_0 = 0$$
$$y_0 = 1$$
$$x_1 = 0 + 0.05 = 0.05$$

$y_1 = 1 + 0.05 \times 0 \times 1 = 1$

$y_1 = 1 + 0.025(0 + 0.05 \times 1) = 1.00125$

$x_2 = 0.05 + 0.05 = 0.10$

$y_2 = 1.00125 + 0.05 \times 0.05 \times 1.00125 = 1.003 + 5313$

$y_2 = 1.00125 + 0.025(0.05 \times 1.00125 + 0.10 \times 1.00375313) = 1.00501095$

$x_3 = 0.10 + 0.05 = 0.15$

$y_3 = 1.00501095 + 0.05 \times 0.10 \times 1.00501095 = 1.01003600$

$y_3 = 1.00501095 + 0.25(0.10 \times 1.00501095 + 0.15 \times 1.01003600) = 1.01131111$

Tabulating the results and calculating the % relative error for the trapezoidal method

n	x_n	True value of $y_n\,(8.S.F.)$	y_n by Euler's method	% relative error in Euler's method	y_n by trapezoidal method	% relative error on trapezoidal method
0	0	1	1	-	1	-
1	0.05	1.00125078	1	1.25	1.00125	0.0000078
2	0.10	1.00501252	1.0025	2.50	1.00501095	0.000156
3	0.15	1.01131352	1.00500625	6.54	1.01131111	0.000024

The switch to the Runge-Kutta method has improved to the accuracy considerably.

The Runge-Kutta method can be shown to have local truncation error of the order h^2.