Parallel and Perpendicular Axis Theorems

Parallel Axis Theorem

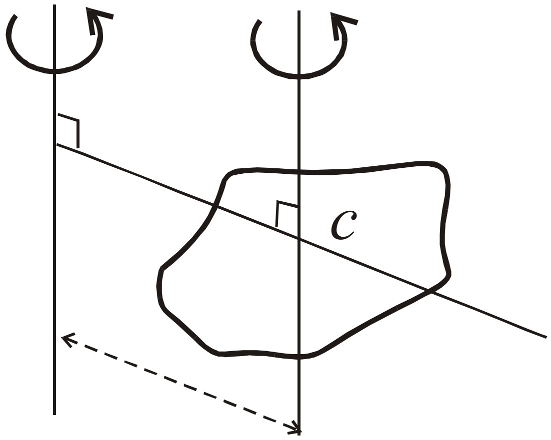
We have obtained standard results for the moment of inertia for certain bodies about an axis passing through each body's centre of mass.

We now need to find the moment of inertia of a body about an axis at a distance *d* from the first axis. Rather than working from first principals every time we can employ the parallel axis theorem. We proceed to state this theorem, illustrate its use and then prove it.

<u>Parallel Axis Theorem</u>

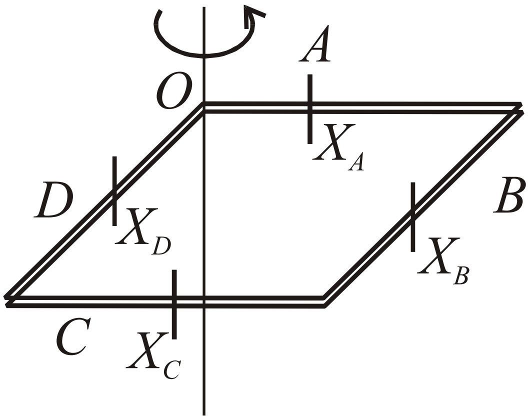
Suppose the moment of inertia of a body M about an axis passing through its centre of mass is Mk^2, then its moment of inertia about an axis parallel to this first axis but at a distance d from it is $M\left(k^2 + d^2\right)$.

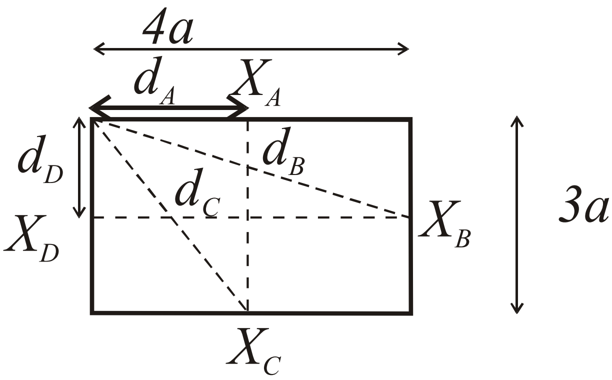
$I = M\left(k^2 + d^2\right)$ $\quad$ $I = Mk^2$



<u>Example (1)</u>

A rectangular grid is made of four thin uniform rods. The rectangle has length $4a$ and width $3a$. The longer pieces have mass $4m$ and the shorter pieces have mass $3m$. Find the moment of inertia of the grid about an axis through one of its corners and perpendicular to its plane.





Label the four pieces *A, B, C, D* with centres of mass X_A, X_B, X_C, X_D respectively.

Then $M_A = 4m, M_B = 3m, M_C = 4m, M_D = 3m.$

About X_A, X_B, X_C, X_D moments of inertia are

$$I_A = I_C = \frac{Ml^2}{3} = \frac{4m(2a)^2}{3} = 4m\left(\frac{2a}{\sqrt{3}}\right)^2$$

$$I_B = I_D = \frac{Ml^2}{3} = \frac{3m\left(\frac{3}{2}a^2\right)}{3} = 3m\left(\frac{3a}{2\sqrt{3}}\right)^2 = 3m\left(\frac{\sqrt{3}a}{2}\right)^2$$

$$d_A = 2a$$

$$d_B = \sqrt{\left(\frac{3}{2}a\right)^2 + (4a)^2} = \sqrt{\left(\frac{9+32}{2}\right)a^2} = \sqrt{\frac{41}{2}}a$$

$$d_C = \sqrt{\frac{41}{2}}a$$

$$d_D = 2a$$

By the parallel axis theorem

$$I_A \text{ about O } = I_A' = M\left(k^2+d^2\right) = 4m\left(\frac{4a^2}{3}+4a^2\right) = \frac{64ma^2}{3}$$

$$I_B \text{ about O } = I_B' = M\left(k^2+d^2\right) = 3m\left(\frac{3a^2}{4}+\frac{41a^2}{2}\right) = \frac{255}{4}ma^2$$

$$I_C \text{ about O } = I_C' = M\left(k^2+d^2\right) = 4m\left(\frac{4a^2}{3}+\frac{41}{2}a^2\right) = \frac{262}{3}ma^2$$

$$I_D \text{ about O } = I_D' = M\left(k^2+d^2\right) = 3m\left(\frac{3a^2}{4}+4a^2\right) = \frac{57ma^2}{4}$$

Therefore

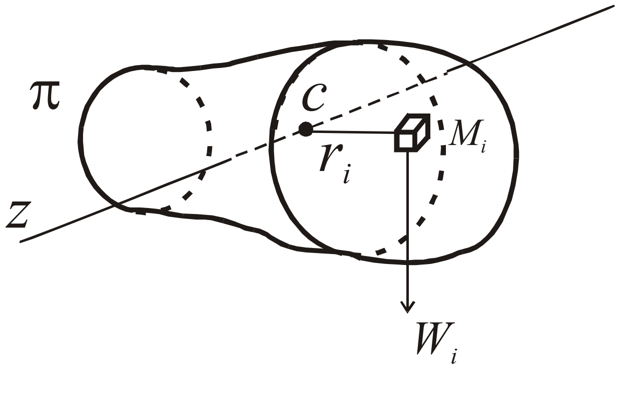
$$\begin{aligned} I_A'+I_B'+I_C'+I_D' &= \left(\frac{64}{3}+\frac{255}{4}+\frac{262}{3}+\frac{57}{4}\right)ma^2 \\ &= \left(\frac{326}{3}+78\right)ma^2 \\ &= \left(\frac{326+234}{3}\right)ma^2 \\ &= \frac{560}{3}ma^2 \end{aligned}$$

We now proceed to prove the parallel axis theorem.

Proof of the parallel axis theorem

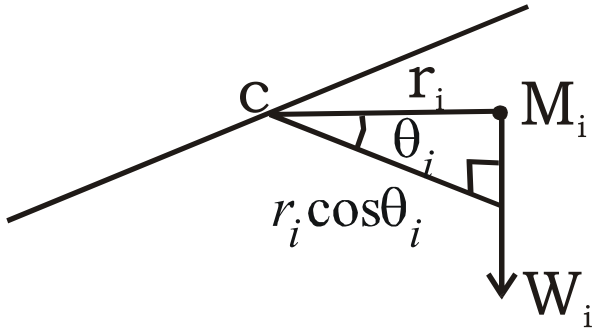
Let π be a body (rod, lamina or solid) of mass M with moment of inertia $I = Mk^2$ about a perpendicular axis z passing through its centre of mass.

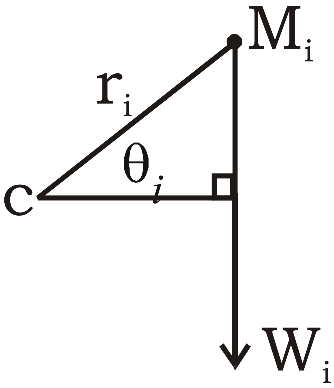
Divide π into small segments each of mass M. Let r_i be the distance of the mass segment M_i from C, the centre of mass of π.



Consider the weight W_i of the mass segment M_i. The torque produced by this weight about the axis z is

$$c_i = w_i r_i \cos\theta_i$$





Here θ_i is the angle made between the perpendicular from W_i to C and the line joining C to M_i and $r_i \cos\theta_i$ s is the perpendicular distance. That is

torque = force × perpendicular distance

results in

$$\begin{aligned} c_i &= w_i r_i \cos\theta_i \\ &= M_i g r_i \cos\theta_i \\ &= g\left(M_i r_i \cos\theta_i\right) \end{aligned}$$

But C is the centre of mass, so the sum of all the moments of the mass elements about this centre of mass is equal to 0. i.e.

$$g\sum m_i r_i \cos\theta_i = 0$$

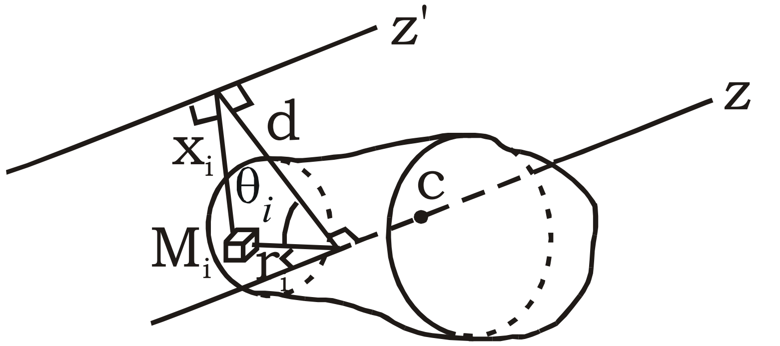
Hence

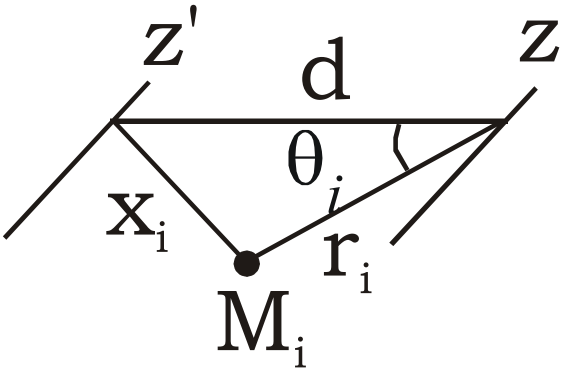
$$\sum m_i r_i \cos\theta_i = 0$$

We are given that the moment of inertia of π about z is Mk^2. In terms of our mass elements M_i, distance r_i from c this means

$$Mk^2 = \sum m_i r_i^2$$

Now consider the moment of inertia of π about another axis l' with perpendicular distance d from z.





We consider a mass element of M_i of π which has perpendicular distance r_i from z and x_i from z'. The moments of inertia of π about z' is approximately the sum of the moments of inertia of all the elements of π

$$I' \approx \sum m_i x_i^2$$

In the limit, as δv_i, the volume of m_i, tends to O, the approximation becomes exact

$$I' = \sum m_i x_i^2 \quad \text{when } \delta v_i \to O$$

But $x_i^2 = d^2 + r_i^2 - 2dr_i \cos\theta_i$

Thus

$$\begin{aligned} I' &= \sum m_i x_i^2 \\ &= \sum m_i \left(d^2 + r_i^2 - 2dr_i \cos\theta_i \right) \\ &= d^2 \sum m_i + \sum m_i r_i^2 - 2d \sum m_i r_i \cos\theta_i \end{aligned}$$

But

$$\sum m_i r_i \cos\theta_i = 0$$
$$\sum m_i = M \text{ the whole mass}$$
$$\sum m_i r_i^2 = Mk^2$$

Thus

$$I' = Md^2 + Mk^2$$
$$I' = M\left(d^2 + k^2\right)$$

Perpendicular Axis Theorem

We will state the perpendicular axis theorem, illustrate it, and then finally prove it. It is best understood through illustration.

Perpendicular Axis Theorem

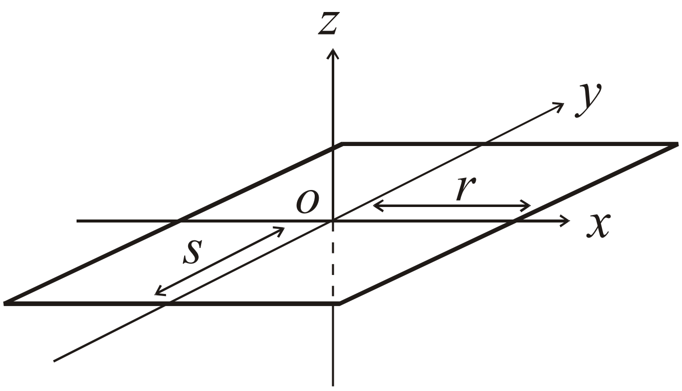
Let *Ox* and *Oy* be two perpendicular axes. Let the moment of inertia of a plane body about Ox be I_x and the moment of inertia of the same plane body about *Oy* be I_y.

Let *Oz* be the axis that is perpendicular to both I_x and I_y. Then the moment of inertia I_z of the body about *Oz* is

$$I_z = I_x + I_y$$

Example (1)

A plane rectangular lamina of mass *M* has length 6m and breadth 4m. Find its moment of inertia about an axis of rotation passing through its centre of mass and perpendicular to the plane.



Standard results give us

$$I_x = \frac{Ml^2}{3} = \frac{M3^2}{3} = 3M$$
$$I_y = \frac{Ml^2}{3} = \frac{M2^2}{3} = \frac{4M}{3}$$

Hence

$$I_z = I_x + I_y = 3M + \frac{4M}{3} = \frac{13M}{3}$$

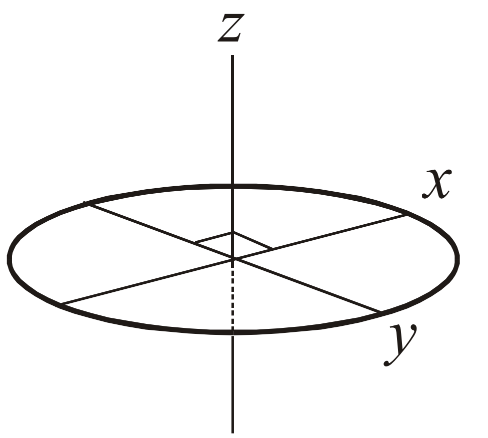
<u>Example (2)</u>

Find the moment of inertia of a thin uniform disc of radius *z* and mass *M* about a diameter.

The standard result gives us that a disc of radius *z* and mass *M* has moment of inertia

$$I = \frac{1}{2}Ml^2$$

through an axis *z* passing through its centre and perpendicular to its plane:



Let *x* and *y* be any two distinct diameters of this disc perpendicular to each other. Since they are diameters they are each other. Since they are diameters they are each perpendicular to *z*. Then by the perpendicular axis theorem

$$I = I_x + I_y$$

But by symmetry,

$I_x = I_y$

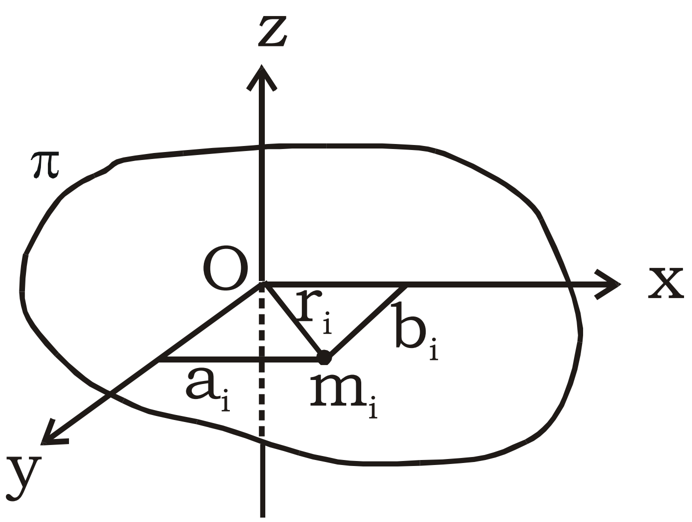
hence

$I = 2I_x$

$$\therefore I_x = \frac{1}{2}I = \frac{1}{2} \times \frac{1}{2} Ml^2 = \frac{Ml^2}{4}$$

Proof of the Perpendicular Axis Theorem

Let π be a plane thin uniform lamina. Let m_i be a mass element with perpendicular distance r_i from an axis *Oz* perpendicular to the plane and passing through *O* in the plane.



Let *Ox* and *Oy* be two perpendicular axes lying in the plane. Let a_i be the perpendicular lying in the plane. Let a_i be the perpendicular distance of m_i from *Ox* and b_i be the perpendicular distance of m_i from *Oy*. Let

$$I_x = \sum m_i a_i^2$$

be the moment of inertia of π about *Ox* and

$$I_y = \sum m_i d_i^2$$

be the moment of inertia of π about *Oy*.

The moment of inertia of π about *Oz* is given by

$$
\begin{aligned}
I_z &= \sum m_i r_i^2 \\
&= \sum m_i \left(a_i^2 + b_i^2\right) && \text{by Pythagoras} \\
&= \sum m_i a_i^2 + \sum m_i b_i^2 \\
&= I_x + I_y
\end{aligned}
$$