Parametric Equations

Curves mapped out by parameters

A curve can be defined in terms of two functions that give the x and y coordinates of a point on that curve in terms of another variable, say t, called a parameter.

Example (1)

Two coordinate functions are defined by

$x(t) = \sqrt{2}\cos t$ $\qquad$ $y(t) = \sqrt{2}\sin t$

where t is a parameter.

(*a*) Complete the following table for arguments of the parameter t and the corresponding values of x and y.

t	0	45	90	135	180	225	270	315	360
$x(t) = \sqrt{2}\cos t$	$\sqrt{2}$	1							
$y(t) = \sqrt{2}\sin t$	0	1							

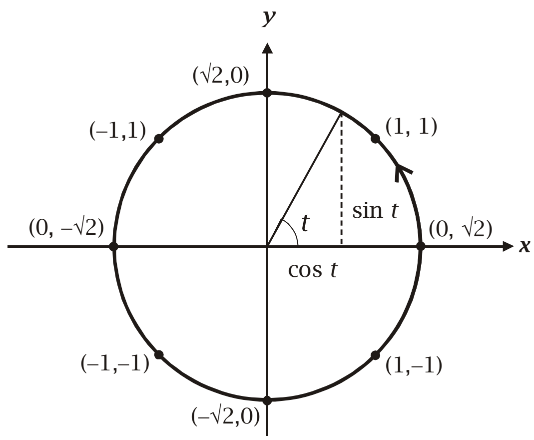
(*b*) Using the data from the table in part (*a*) sketch the locus of the point (x, y) as it varies with t. State what the shape is and interpret the meaning of t.

Solution

(*a*)

t	0	45	90	135	180	225	270	315	360
$x(t) = \sqrt{2}\cos t$	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$
$y(t) = \sqrt{2}\sin t$	0	1	$\sqrt{2}$	1	0	-1	$\sqrt{2}$	-1	0

(*b*) The parameter t may be interpreted as giving the angle of a circle.



A *parameter* is a variable on which one or more other variables depend. Often the parameter represents time, but this is not always the case. In example (1)

$x(t) = \sqrt{2}\cos t \qquad\qquad y(t) = \sqrt{2}\sin t$

the parameter represented the angle of a circle. We can eliminate t from these expressions to obtain the Cartesian equation for this circle.

$$x = \sqrt{2}\cos t \qquad y = \sqrt{2}\sin t$$
$$x^2 = 2\cos^2 t \qquad y^2 = 2\sin^2 t$$
$$x^2 + y^2 = 2\left(\cos^2 t + \sin^2 t\right)$$
$$x^2 + y^2 = 2 \qquad\qquad \text{Since } \sin^2 t + \cos^2 t = 1$$

This confirms that the curve is a circle with radius $\sqrt{2}$.

Example (2)

A curve is defined by the parametric equations

$x = t + 2 \qquad\qquad y = t^2 - 2$

(*a*) By eliminating t from both expressions obtain a relationship between x and y.

(*b*) Use the Cartesian equation found in part (*a*) to sketch the curve represented by these parametric equations.

Solution

(*a*) $x = t + 2 \qquad (1)$

$y = t^2 - 2 \qquad (2)$

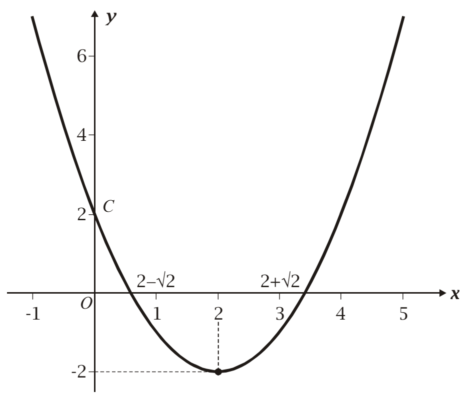
From (1)

$t = x - 2$

Substituting in (2)

$y = (x-2)^2 - 2$

(*b*) This is the equation of a parabola, with the minimum point at $x = 2$, $y = -2$. This parabola cuts the *x*-axis where $x = 2 \pm \sqrt{2}$. The parabola intercepts the *y*-axis where $x = 0$ and $y = 2$.



The gradient of a curve defined parametrically

The gradient of such a curve defined parametrically is found from an application of the chain rule

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Example (3)

Find the gradient at the point (x, y) of the circle defined parametrically by

$x(t) = \cos t \qquad y(t) = \sin t$

Solution

The gradient is $\frac{dy}{dx}$ but we first begin by differentiating each coordinate function with respect to the parameter *t*.

$$\frac{dy}{dt} = \cos t$$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$$

Example (4)

Given that $x(t) = 1 + t^4$, $y(t) = 1 - t^2$, find $\frac{dy}{dx}$ in terms of *t*.

Solution

$$\frac{dy}{dt} = 4t^3$$

$$\frac{dx}{dt} = -2t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{-2t} = -2t^2$$

Example (5) [WJEC June 2002]

A curve has parametric equations $x = at,\ y = \frac{a}{t}$ where *a* is a non-zero constant.

(*a*) Show that the tangent to the curve at the point *P*, whose parameter is *p*, has equation $p^2y + x = 2ap$.

(*b*) This tangent intersects the *x*-axis and the *y*-axis at *Q* and *R*, respectively. Show that *P* is the mid-point of *QR*.

Solution

(*a*) $x = at \quad \Rightarrow \quad \frac{dx}{dt} = a$

$y = \frac{a}{t} \quad \Rightarrow \quad \frac{dy}{dt} = -at^{-2} = -a\frac{1}{t^2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{1}{t^2}$$

At the point P the gradient of the tangent is $-\frac{1}{p^2}$.

The point P has coordinates $(x_p, y_p) = \left(ap, \frac{a}{p}\right)$

On substitution into $y - y_p = m(x - x_p)$

$$y - \frac{a}{p} = -\frac{1}{p^2}(x - ap)$$

$$p^2y - ap = -x + ap$$

$$p^2y + x = 2ap$$

(*b*) The coordinates of Q are given by subsituting $y = 0$ into this equation $p^2 y + x = 2ap$

$y = 0 \quad \Rightarrow \quad x = 2ap \quad \Rightarrow \quad Q = (2ap, 0)$

Likewise

$x = 0 \quad \Rightarrow \quad y = \frac{2a}{p} \quad \Rightarrow \quad R = \left(0, \frac{2a}{p}\right)$

We have also $P = \left(ap, \frac{a}{p}\right)$

$$|PQ| = \sqrt{(2ap - ap)^2 + \left(0 - \frac{a}{p}\right)^2} = a\sqrt{p^2 + \frac{1}{p^2}}$$

$$|PR| = \sqrt{(ap - 0)^2 + \left(\frac{a}{p} - \frac{2a}{p}\right)^2} = a\sqrt{p^2 + \frac{1}{p^2}}$$

Hence

$|PQ| = |PR|$