Pascal's Triangle and the Binomial Theorem

Prerequisites

You should be able to expand expressions like $(1+x)^3$.

Example (1)
Find the values of a and b in
$(1+x)^3 = 1 + ax + bx^2 + x^3$

Solution

$$\begin{aligned}
(1+x)^3 &= (1+x)(1+x)(1+x) \\
&= (1+x)(1+2x+x^2) \\
&= 1+2x+x^2+x+2x^2+x^3 \\
&= 1+(1+2)x+(1+2)x^2+x^3 \\
&= 1+3x+3x^2+x^3
\end{aligned}$$

Hence $a = b = 3$

This method of expanding $(1+x)^3 = (1+x)(1+x)(1+x)$ by multiplying out terms involves a lot of algebra, and for that reason is also prone to human error. The problems with this method increase as the number of terms increases. For example, finding $(1+x)^9$ would be quite a tedious process by this method. What is required is a short cut from $(1+x)^n$ to its expansion that is valid for any *n*. This is provided by the Binomial Theorem.

Binomial products

Expressions involving brackets that contain two terms are called *binomial products*. They are called binomial because there are *two* terms in each bracket. Here we are concerned with binomial products that take the form

$(a+b)^n$

where n is a positive integer and a and b are real numbers. When binomial products of this form are expanded on a term-by-term basis we call that a *binomial expansion*. The binomial expansion of $(a+b)^3$ is

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

which you can confirm by multiplying out the terms if you wish.

Example (2)

You are given

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The second term of this expansion is $4a^3b$. This term comprises the product of the coefficient (number) 4 with the term a^3 and the term b. The exponent of a^3 is 3. The exponent of $b = b^1$ is 1.

(*a*) Observe the exponents of a in successive terms. What pattern do these exhibit?

(*b*) Likewise, what is the pattern governing the exponent of b in successive terms?

(*c*) It what ways are the coefficients in $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ symmetrical?

Solution

(*a*) Regarding the exponent of a each successive term of $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ is 1 less than that of the preceding term.

a^4 ... a^3 ... a^2 ... a^1 ... a^0

Recall that in the above $a^0 = 1$.

(*b*) The exponent of b exhibits the same pattern as (*a*) but only in reverse; the exponent is increased by one for each successive term.

b^0 ... b^1 ... b^2 ... b^3 ... b^4

(*c*) There are five terms each with a coefficient. The coefficients form a symmetrical pattern where the first coefficient is the same as the last, and the second is the same as the fourth.

1 ... 4 ... 6 ... 4 ... 1

All binomial expansions of the type $(a+b)^n$ exhibit the same symmetry properties. The exponent of the leading term a decreases by 1 with each successive term; the exponent of the second term b increases by 1 with each successive term, and the coefficients exhibit a symmetrical pattern.

Example (3)

(a) Given that the coefficients of $(a+b)^5$ are

1 ... 5 ... 10 ... 10 ... 5 ... 1

find its binomial expansion.

(*b*) Find the binomial expansion of $(2+x)^5$.

Solution

(a) $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

(b) Here $a = 2$ and $b = x$, so

$$\begin{aligned}(2+x)^5 &= 2^5 + 5(2)^4 x + 10(2)^3 x^2 + 10(2)^2 x^3 + 5(2)^1 x^4 + 1(2)^0 x^5 \\ &= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5\end{aligned}$$

Pascal's triangle

The previous examples show that in order to expand *any* binomial product of the form $(a+b)^n$ all we really need to know are the coefficients that are inserted into the series, for otherwise the powers of successive terms in a descend and the powers of successive terms in b ascend. Furthermore, by substituting other numbers or symbols for a and b, any binomial expansion can be found. There is a short-cut process for finding these *binomial coefficients* that is known as *Pascal's triangle*. The binomial coefficients can be arranged into a symmetrical triangle. Here are the first three rows of this triangle.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

There is a rule for generating successive rows, which this diagram illustrates.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & \searrow & 2 \swarrow & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
$$

$$1 + 2 = 3$$

In this triangle the numbers are generated according the rule that the entries in one row arise from adding the two diagonal elements immediate to the left and right of it in the row above it.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Example (7)

Use Pascal's triangle to find the expansion of

(a) $(1-2x)^5$

(b) $(a+b)^7$

Solution

(a) The coefficients of the 5th row of Pascal's triangle are

1, 5, 10, 10, 5, 1

This gives

$$
\begin{aligned}
(1-2x)^5 &= \left(1 \times 1^5 \times (-2x)^0\right) + \left(5 \times 1^4 \times (-2x)^1\right) + \left(10 \times 1^3 \times (-2x)^2\right) \\
&\quad + \left(10 \times 1^2 \times (-2x)^3\right) + \left(5 \times 1^1 \times (-2x)^4\right) + \left(1 \times 1^0 \times (-2x)^5\right) \\
&= 1 - (5 \times 2x) + \left(10 \times 4x^2\right) - \left(10 \times 8x^3\right) + \left(5 \times 16x^4\right) - \left(1 \times 32x^5\right) \\
&= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5
\end{aligned}
$$

(b) We must first generate the 7th row of Pascal's triangle by adding diagonal elements from the 6th row.

$$
\begin{array}{ccccccccccccccc}
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Then

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

Binomial coefficients

The coefficients in the expansion of the expression $(a+b)^n$ are called binomial coefficients. They are denoted by

${}^nC_r$ or $\binom{n}{r}$

The symbol ${}^nC_r = \binom{n}{r}$ means *the* r*th coefficient of the expansion of* $(a+b)^n$ with the rule that we start with $r = 0$. For example, in

$$(a+b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

we have

$$\binom{4}{0}=1 \quad \binom{4}{1}=4 \quad \binom{4}{2}=6 \quad \binom{4}{3}=4 \quad \binom{4}{4}=1$$

There is a formula for the value of the coefficient $\binom{n}{r}$.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

In this expression the factorial symbol $(!)$ stands for

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

and $0! = 1$ by definition. Using this formula it is not necessary to generate the whole of Pascal's triangle to find a given coefficient in a binomial expansion.

Example (8)

Find $\binom{10}{3}$.

Solution

$$\binom{10}{3} = \frac{10!}{3!\,(10-3)!} = \frac{10!}{3!7!} = 120$$

Electronic calculators generally have a button for finding the binomial coefficient. Recall how in example (7) we created the seventh row of Pascal's triangle by adding the diagonal elements of the row above it.

$$\begin{array}{ccccccccccccccc}
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow\swarrow & & \searrow \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

We can write this rule for generating Pascal's triangle using binomial coefficients.

$$\begin{tikzcd}
\binom{n-1}{r-1} \arrow[dr] & & \binom{n-1}{r} \arrow[dl] \\
 & \binom{n}{r} &
\end{tikzcd}$$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

The Binomial Theorem

The use of binomial coefficients to expand binomial products is justified by the Binomial Theorem. This theorem states that the expansion of $(a+b)^n$ is given by

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n-r}a^{n-r}b^r + ... + \binom{n}{n}b^n$$

The proof is by Mathematical Induction, which is a technique introduced at a higher level. Consequently, at this stage the student has to take the formula on trust.[1]

Example (9)

Find the first three terms and the last term in the expansion of $(1+3x)^9$ as a series of ascending powers of x.

Solution

[1] If you are interested there is a separate chapter on this topic at our database entitled *Proof of the Binomial Theorem*

$$(1+3x)^9 = 1 + \binom{9}{1}(3x) + \binom{9}{2}(3x)^2 + \ldots + \binom{9}{9}(3x)^9$$

$$\binom{9}{1} = \frac{9!}{1!(9-8)!} = \frac{9!}{8!} = 9$$

$$\binom{9}{1}(3x) = 9 \times 3x = 27x$$

$$\binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{9!}{2!7!} = 36$$

$$\binom{9}{2}(3x)^2 = 36 \times 9x^2 = 324x^2$$

$$\binom{9}{9} = \frac{9!}{9!(9-9)!} = \frac{9!}{9!0!} = 1$$

$$\binom{9}{9}(3x)^9 = 1 \times 3^9 x^9 = 19683x^9$$

Hence

$$\begin{aligned}(1+3x)^9 &= 1 + \binom{9}{1}(3x) + \binom{9}{2}(3x)^2 + \ldots + \binom{9}{9}(3x)^9 \\ &= 1 + 27x + 324x^2 + \ldots + 19683x^9\end{aligned}$$

The binomial coefficient can be used to find the coefficients in large binomial expansions.

Example (10)

Find the coefficient of the x^7y^8 term in the expansion of $(3x+2y)^{15}$.

Solution

$$(3x+2y)^{15} = \binom{15}{0}(3x)^{15}(2y)^0 + \binom{15}{1}(3x)^{14}(2y)^1 + \ldots + \binom{15}{8}(3x)^7(2y)^8 + \ldots$$

The term we are interested in is

$$\binom{15}{8}(3x)^7(2y)^8$$

The binomial coefficient is evaluated as

$$\binom{15}{8} = \frac{15!}{8!(15-8)!} = \frac{15!}{8!7!} = 6435$$

$$\begin{aligned}\binom{15}{8}(3x)^7(2y)^8 &= 6435 \times 3^7x^7 \times 2^8y^8 \\ &= 3{,}602{,}776{,}320\end{aligned}$$

You can see why you might not want to write out Pascal's triangle for this question!

Example (11)

In the binomial expansion of $(a+2x)^5$, the coefficient of the term in x^2 is equal to the coefficient of the term in x^3. Given that $a \neq 0$, find the value of *a*.

Solution

From Pascal's triangle the binomial coefficients for the fifth row are 1, 5, 10, 10, 5, 1. So the expansion of $(a+2x)^5$ is

$$\begin{aligned}(a+2x)^5 &= a^5 + 5a^4(2x) + 10a^3(2x)^2 + 10a^2(2x)^3 + \ldots \\ &= a^5 + 10a^4x + 40a^3x^2 + 80a^2x^3 + \ldots\end{aligned}$$

The condition in the question implies

$$\begin{aligned}40a^3 &= 80a^2 \\ 40a &= 80 \quad (\text{since } a \neq 0) \\ a &= 2\end{aligned}$$