# Permutation groups

## Permutations

A permutation is an arrangement of numbers or objects in a specific order. For example, there are six permutations of the numbers, 1,2 and 3 as follows

$(1,2,3)$
$(1,3,2)$
$(2,1,3)$
$(2,3,1)$
$(3,1,2)$
$(3,2,1)$

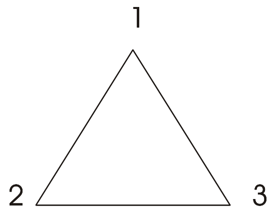
Group theory deals with the structures created when a binary operation is defined on a set in such a way as to satisfy the group structure properties of (1) existence of identities, (2) existence of inverses; (3) closure of the group; (4) associativity of the operation.

An example of a group is the symmetry group of the equilateral triangle, which shall be discussed below. There are six transformations that can be described as acting on an equilateral triangle so that the triangle is left unchanged at the end of it – these are basically reflections in certain axes and certain rotations about the centre. These transformations form a set, and the binary operation of composition of transformations makes that set into a group. Hence the symmetry group of the equilateral triangle.

However, mathematicians are always looking for ways of turning one structure into another. It is useful to manipulate numbers rather than transformations. One approach is to label the vertices of, in this case, the equilateral triangle, and to examine how the various transformations of the symmetry group affect these triangles. This is only an example, but it leads us to the idea in general of a permutation group – that is a group defined on a set of numbers, made up of the collection of all permutations of those numbers. Let us examine in detail how this works for the symmetry group of the equilateral triangle.

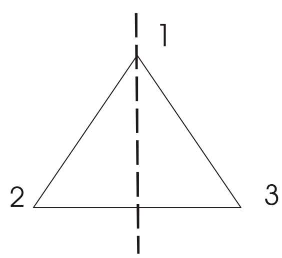
## Symmetry groups are also permutation groups

The group of symmetries of an equilateral triangle is an example of a permutation group. If we label the vertices of an equilateral triangle, we can see that each symmetry can be regarded as a rearrangement of these labels - that is, as a permutation.



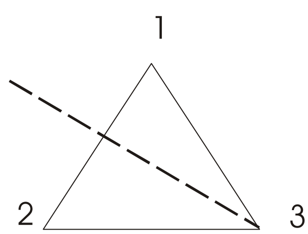

Identity, I

$1 \to 1$ The identity does nothing to the triangle
$2 \to 2$
$3 \to 3$



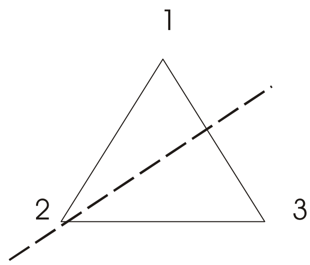

Reflection, $Q_1$

$1 \to 1$ This reflection swops the vertices 2 and 3
$2 \to 3$
$3 \to 2$




Reflection, $Q_2$

$1 \to 2$ This reflection swops the vertices 1 and 2
$2 \to 1$
$3 \to 3$

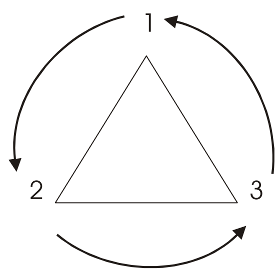



Reflection, $Q_3$

$1 \rightarrow 3$ This reflections swops the vertices 1 and 3

$2 \rightarrow 2$

$3 \rightarrow 1$

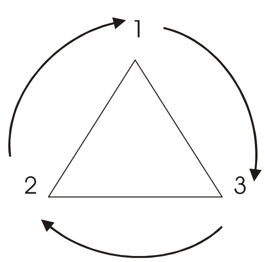



Rotation, $R_1$

$1 \rightarrow 2$ This rotation cylically permutes the

$2 \rightarrow 3$ vertices as indicated

$3 \rightarrow 1$




Rotation, $R_1$

$1 \rightarrow 3$ This rotation cylically permutes the

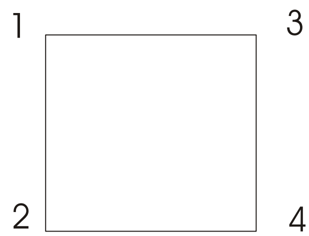
$2 \rightarrow 1$ vertices in the opposite direction

$3 \rightarrow 2$

As the example illustrates a permutation of the symbols 1,2,3,… is a one-one mapping (a bijection) from the set {1,2,3,…} to itself.

**Symmetry group of the square**

The symmetries of a square are the permutations of the four symbols 1,2,3,4.



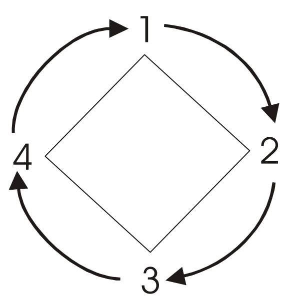

Permutation groups are represented by the symbol S,

For example, $S_4$ is the permutation group of 4 symbols.

As the above example for the equilateral triangle shows, each permutation can be represented by a mapping diagram.

<u>Anticlockwise rotation of a square through</u> $\pi/2$

As another example, rotation anticlockwise of a square through $\pi/2$ is equivalent to the mapping




$1 \to 2$

$2 \to 3$

$3 \to 4$

$4 \to 1$

A more convenient notation is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

A yet further way of representing a permutation is by observing, for example, here that

1➔2➔3➔4➔1

This means 1 maps to 2, 2 maps to 3, 3 maps to 4, 4 maps to 1.

If we loop the 4 back to the 1 then this cycle can be simply written as

$(1,2,3,4)$

That is, besides indicating the following cycle

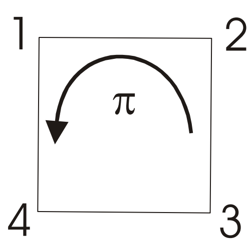
1➔2➔3➔4

this symbol means that the 4 is followed by the 1

4➔1

This is another convention for writing permutations

<u>Rotation through $\pi$</u>

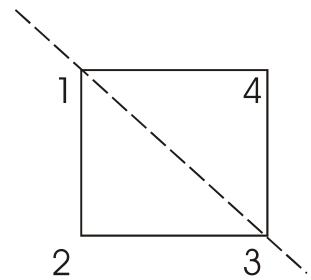



Corresponds to two cycles:


Hence, the cyclical representation is

$(1,3)(2,4)$

Reflections sometimes leave some vertices unmoved. In this case, in the cyclical representation the unchanged vertices are simply omitted. It is assumed that if a vertex is not indicated as part of a cycle that it is not moved. For example:

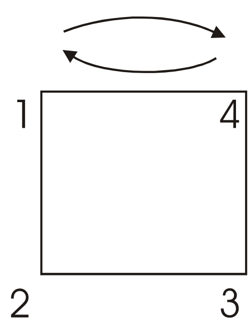



This reflection cyclically permutes the vertices 2 and 4 leaving vertices 1 and 3 unchanged, and hence has cyclical representation

(2 4)

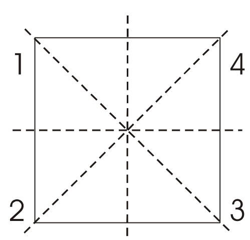
This discussion of the symmetries of the square in the context of permutation group $S_4$ makes it clear that the group of all symmetries of the square is a subgroup of the permutation group $S_4$.

For example - the permutation (1 4) is not a symmetry of the square




It would involve twisting the square in a way that is not acceptable as a symmetry.

There are 4! = 24 permutations of 4 symbols, but there are 8 symmetries of a square.




The symmetries of a square comprise four reflections and four rotations. The four rotations include the identity symmetry.