Permutations and Combinations

Introduction

The topic of permutations and combinations answers questions like

> In how may ways can eight different books be arranged on a shelf?
>
> In how many ways can the management of a tennis club arrange a team of four mixed pairs when there are nine men and seven women to choose from?

There are techniques that assist in the solution of such questions.

Permutations of *n* different items

A *permutation* is an ordered arrangement of items in a set. Suppose, for example, we have four books, A, B, C and D, which are to be arranged on a shelf. We treat any different order of the books (from left to right) as a different arrangement. In that case, there are 24 different arrangements (permutations), which can be listed systematically as follows.

ABCD	BACD	CABD	DABC
ABDC	BADC	CADB	DACB
ACBD	BCAD	CBAD	DBAC
ACDB	BCDA	CBDA	DBCA
ADBC	BDAC	CDAB	DCAB
ADCB	BDCA	CDBA	DCBA

Let us explain how we systematically make such a list of the items ordered in a list, using the items A B C D as an example. Here we start with four items. After choosing the first item (for example, starting with A), there are three left. In order to list all the permutations of these three, we take each one of these items in order and list all possible orderings of the items (BCD) that remain. The process is repeated (iterated) until there is only one item to choose from.

A	B	CD
		DC

To further illustrate this process, let us take the first of these items, here A, and list all the possible permutations of the remaining items.

A	BCD
	BDC
	CBD
	CDB
	DBC
	DCB

When this has been done, repeat the process for the second of the items in the list.

B	ACD
	ADC
	CAD
	CDA
	DAC
	DCA

and so on, until all the items have been listed. In many practical applications, we simply wish to know the number of permutations and an actual list is not required. Therefore, listing all the permutations is a waste of energy. To find the number of permutations of n items ordered in a list, we argue as follows.

Suppose there are n items in a list.
Then there are n ways of choosing the first item.
After that item has been removed there are $n-1$ items.
So, there are $n - 1$ ways of choosing the next item, leaving $n - 2$ items.
So, there are $n - 2$ ways of choosing the next item, leaving $n - 3$ items.
And so on …

Permutations of *n* different items

There are $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ permutations of a set of *n* elements.

Example (1)

In how may ways can eight different books be arranged on a shelf?

Solution

Number of permutations = 8! = 40320

Permutations of *n* different items taken *r* at a time

This is an arrangement of *r* objects taken from a set of *n* objects.

Example (2)

In how many ways can 4 objects be arranged when there are 6 objects to choose from?

Solution

To solve this problem we reason as follows

For the first choice there are 6 objects to choose from.
Once the first choice has been removed, there are 5 objects to choose from.
Once the second choice has been removed, there are 4 objects to choose from.
Finally, there are 3 objects to choose from for the fourth choice.

Therefore, there are

$6 \times 5 \times 4 \times 3 = 360$

permutations of 4 objects chosen from a set of 6 items.

We use the symbol ${}^{n}P_{r}$ to denote choosing *r* items form a set of *n* elements.

Example (2) continued

The solution may be written simply as

${}^{6}P_{4} = 6 \times 5 \times 4 \times 3 = 360$

Calculations involving large numbers, say 15 items chosen from a set of 153 items can be tedious. We want a short cut. This is provided by the formula

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

To illustrate this formula

$${}^{6}P_{4} = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{720}{2} = 360$$

To show why this works, we may expand the factorial symbols, and cancel

$${}^{6}P_{4} = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times2\times1}{2\times1} = \frac{6\times5\times4\times3\times\not{2}\times\not{1}}{\not{2}\times\not{1}} = 6\times5\times4\times3 = 360$$

So we obtain the same formula as before. The formula is useful when calculating large permutations. However, most calculators have a button that permits the formula to be evaluated without recourse to the factorial symbol. But it is still useful to know how to demonstrate the validity of the formula.

Permutations with "identical" twins

Another problem is illustrated by the following

> How many arrangements of 10 books are there on a shelf, if 9 of the books are different but the tenth is a copy of one of the others?

Let us first tackle a similar problem based on a smaller set – four items, two of which are identical. Thus, we have a set consisting of

A B C C

We wish to find the number of arrangements of these four items. To explain the solution, let us suppose that the two "twins" can be marked in some way to distinguish the one from the other. Let us use an asterisk for this purpose.

A B C C*

Now, we argue as before; there are four ways of choosing the first element, three ways of the choosing the second element, and so forth, giving 4! = 24 permutations. But, when we list these permutations, we are reminded that some of them shall count as the same permutation. For example

A	C	C*	B
A	C*	C	B

are counted the same permutation, because C and C* are "twins". Likewise

A	C	B	C*
A	C*	B	C

are the same permutation. To further illustrate this point, suppose now we have four items, but three of them are identical.

A	B	B	B

We will mark the identical elements to distinguish them

A	B	B*	B**

Now, let us list the permutations of these four now "different" elements. But in that case, some of the permutations shall be regarded as identical because B, B* and B** are "twins" ; for example

A	B	B*	B**
A	B	B**	B*
A	B*	B	B**
A	B*	B**	B
A	B**	B	B*
A	B**	B*	B

are all identical permutations. The total number of permutations must be divided by the number of permutations of the 3 identical elements – that is by 3! = 6. Thus, for every identical pair we have to divide the total number of permutations by 2. Likewise, when there are three identical elements we divide the total number of permutations by 3! = 6, and the general rule is as follows.

The number of permutations of *n* items taken *n* at a time, when *p* times are identical

The number of permutations of n items taken n at a time, when p times are identical, the others being different, is $\frac{n!}{p!}$

Example (3)

How many arrangements of 10 books are there on a shelf, if 9 of the books are different but the tenth is a copy of one of the others?

Solution

$n = 10$ and $p = 2$

Number of arrangements $= \frac{n!}{p!} = \frac{10!}{2!} = 1814400$

Two sets of identical twins

With two sets of identical twins, where there are n items, p identical of one kind and q identical of another, the number of permutations is $\frac{n!}{p!q!}$

Example (4)

A child is playing a game in which he is arranging marbles in a row. There are twelve marbles in all, but three are identically blue, and two are identically red. How many different arrangements could the child make, assuming he had the time and patience?

Solution

$n = 10, p = 3, \; q = 2$

Number of arrangements $= \frac{n!}{p!q!} = \frac{12!}{3!2!} = 39{,}916{,}800$

Permutations of objects arranged in a circle

Another type of problem is illustrated by

Example (5)

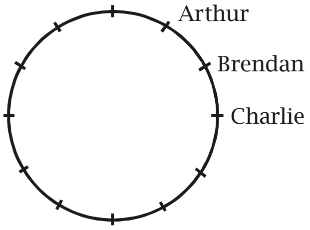
What is the number of different arrangements of twelve people seated at a circular conference table?

One is tempted to say that the answer is 12! – and that may in fact be the answer, but whether it is or not depends on how the question is interpreted. If we regard every seat at the circular table as different, so that if one of the conference members, say Arthur, has a different seat then that will be a different arrangement, then the answer is 12! As before, we argue as follows

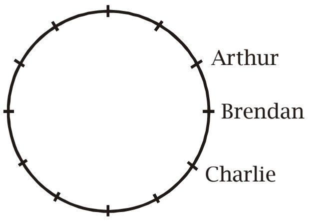
Suppose we label the seats 1 to 12.

Then there are 12 ways of filling the first seat, 11 ways of filling the second seat, and so forth; hence there are 12! permutations.

However, suppose the question is interpreted as asking how many different arrangements of people can be made at the conference table, regardless of which seat the people occupy. In this case the number of permutations is reduced. To see why, consider three of the delegates, Arthur, Brendan and Charlie. Suppose they are seated as follows.



If they all move along one chair, Brendan is still next to both Arthur and Charlie.



If all the delegates to the conference have moved up just one chair, then we have the same arrangement. Since there are 12 seats, we must divide the total number of permutations found on the assumption that the seats were all unique, by 12. So the answer to this question is

$$\text{Number of permutations } = \frac{12!}{12} = 11!$$

Permutations when objects must stay together

Example (6)

Ten children are required to line up at the start of their swimming lesson. But Bert will not be separated from Louise. How many different arrangements are possible?

Solution

To solve this problem we reason as follows.

Firstly, since Bert and Louise cannot be separated, we treat them as a single unit. This means that we are looking for the permutations of nine children or pairs of children. Hence, there are 9! such permutations. However, Bert could be followed by Louise or Louise could be followed by Bert.

– Bert – Louise –

is a different arrangement from

– Louise – Bert –

Hence, we must multiply the number of permutations by 2 in order to find the overall number of permutations

$\text{Number of permutations} = 9! \times 2 = 735760$

Example (7)

How many ways are there of arranging 7 cars in a line in a showroom if 3 of the cars must be placed in a group?

Solution

We treat the three cars initially as a single "block". Hence with the other four cars that can be freely permutated this makes 5! permutations. However, the three cars in the "block" can also be permutated in 3! ways, so the final answer is

$\text{Number of permutations} = 5! \times 3! = 720$

Combinations

A *combination* is simply a set or group of objects with no particular order chosen from a larger set or group. If we are choosing five people from a group of five, then there is clearly only one combination. The question of combinations becomes interesting when the number of items chosen is less than the number that could be chosen.

Example (8)

How many combinations of three people can be made from a group of ten people?

It should be appreciated that the answer to this question is different from the answer to the question about permutations. In a combination the order in which the items are chosen does not matter. Hence

Arthur – Brendan – Charlie

is the same combination as

Brendan – Arthur – Charlie

but a different permutation. Permutations are **ordered**, combinations are **not ordered**.

To solve the problem we reason as follows

There are 10 ways of choosing the first member of the group
9 ways of choosing the second member of the group
8 ways of choosing the third member of the group
This gives the total number of permutations of three members from a set of 10 as
$10 \times 9 \times 8$
However, some of these permutations are identical as combinations, so we must divide this number by the number of permutations of three elements, which is 3!
Hence the answer is

$$\text{Number of combinations} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$$

As before, we have a symbol and a short cut for finding a number of combinations.
The number of combinations of *r* objects chose from *n* different objects is denoted by

$${}^{n}C_{r} \text{ or } \binom{n}{r}$$

And it is computed by the formula

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

which is

$${}^{n}C_{r} = \frac{(\text{total number available})!}{(\text{number chosen})! \times (\text{number not chosen})!}$$

<u>Example (9)</u>

Find the number of combinations when 6 objects are chosen from a set of 11.

Solution

$${}^{n}C_{r} = \frac{11!}{6!(11-6)!} = \frac{11!}{6!5!} = 462$$

Calculators have a function for finding this number. This number is also known as the binomial coefficient, because of the role it plays in the binomial theorem for the expansion of expressions such as $(a+b)^{11}$. The coefficient is also used to compute coefficients to the probability distribution known as the binomial distribution.

Combinations formed from two sets

Example (10)
A committee of 6 people is to be formed from a set of 8 men and 10 women. How many different combinations of members are possible if there are to be 2 men and 4 women?

Solution
We are choosing two men from 8 and 4 women from 10. Since either of the 2 chosen men can be combined with any one of the 4 women, we multiply the combinations to obtain

$$\begin{aligned}\text{number of combinations} &= \begin{array}{l}\text{number of ways}\\ \text{of choosing 2 men}\end{array} \times \begin{array}{l}\text{number of ways}\\ \text{of choosing 4 women}\end{array}\\ &= {}^{8}C_{2} \times {}^{10}C_{4}\\ &= \frac{8!}{2!6!} \times \frac{10!}{4!6!}\\ &= 28 \times 210\\ &= 5880\end{aligned}$$

Combinations where items chosen can be identical

Example (11)
A supermarket has ten varieties of fruit juice. A gentleman wishes to purchase 4 packets, but he wants 2 packets to be the same variety. In how many ways can he make his choice?

Solution
We begin by treating the choice of the 2 identical packets as one choice, to obtain ${}^{10}C_{3}$ combinations of varieties.

However, suppose the gentleman has chosen varieties A, B and C, then he still has to choose which of these varieties will be doubled up. Choosing to double up variety A gives a different overall combination from choosing to double up variety B.

A A B C

is a different combination from

A B B C

and

A B C C

so we must multiply ${}^{10}C_3$ by 3, which is the number of possible choices of which variety to double up, to obtain

$$\text{Number of combinations} = {}^{10}C_3 \times 3 = \frac{10!}{3!7!} \times 3 = 360$$

Harder problems

Harder problems require systematic reasoning!

Example (12)

Twelve boys are to be seated in an examination room in a row. However, three of the boys are naughty and cannot sit next to one another. Find the number of possible arrangements.

Solution

This asks for a number of permutations. Let us label the three naughty boys as X, Y and Z. First of all we start by finding the total number of permutations of the twelve boys – this is 12! Then, we need to subtract all those permutations in which X and Y, X and Z and Y and Z are next to one another. To find the number of permutations in which X and Y are together, treat XY as a single choice. This reduces the set to 11 elements, and hence there are 11! such permutations in which X and Y are next to one another. But the arrangement XY is different from the arrangement YX, so we must multiply this number by 2. Finally, we have to repeat this process for each of X and Y, X and Z and Y and Z. Hence, the number of permutations in which two of the naught boys are together is

$$11! \times 2 \times 3$$

This does not, however, solve the problem. The problem is that in this method we have subtracted certain arrangements twice. For example, when subtracting the number of permutations in which X and Y are together we also subtract the following permutations

X Y Z

Z X Y

That is, we subtract permutations in which Y and Z are together and the permutations in which X and Z are together. In other words, we have subtracted certain permutations more than once. In order to overcome this difficulty we must add back all those permutations in which XY and Z are together. To find this number we reason as follows. Treat XYZ as a single choice. This leaves us with 10 elements, and hence 10! permutations in which XYZ are together in some order. However, we have to consider the alternative arrangements of these elements X, Y and Z. There are 3! of these. Consequently, there are 10! × 3! arrangements in which XY and Z appear together. Finally, the answer is given by

$$\begin{array}{c}\text{number of}\\\text{permuations}\end{array} = \begin{array}{l}\text{total number}\\\text{of permutations}\end{array} - \begin{array}{l}\text{number of permuations}\\\text{in which XY, XZ or YZ}\\\text{appear together}\end{array} + \begin{array}{l}\text{number of permutations}\\\text{in which XYZ appear}\\\text{together}\end{array}$$

$$= 12! - 2 \times 3 \times 11! + 6 \times 10!$$
$$= 261{,}273{,}600$$

Example (13)

A gambler is attempting to forecast football results. He wishes to predict the results for Manchester United over 10 matches. There can be four outcomes: win, lose, score draw, goalless draw. What is the probability that he will gain exactly seven or more correct predictions?

Solution

There is only one way of predicting all ten matches: ${}^{10}C_{10} = 1$.

There are ${}^{10}C_9$ ways of predicting nine correct matches out of ten. However, in that case one of the matches is also incorrectly predicted. There are three ways in which the prediction can fail. Suppose, for instance, the match was a win, then the prediction can fail if it is predicted to be a lose, score draw or goalless draw. Hence there are ${}^{10}C_9 \times 3$ ways of getting nine predictions right out of ten.

There are ${}^{10}C_8$ ways of predicting eight correct matches out of ten. But now two of the matches are incorrectly predicted. There are three ways in which the first incorrect prediction can fail, and three ways in which the second incorrect prediction can fail. Since these predictions are independent of each other there are $3^2 = 3 \times 3$ ways in which the prediction can fail. Hence there are ${}^{10}C_8 \times 3^2$ ways of getting eight predictions right out of ten.

By a similar line of reasoning, there are ${}^{10}C_8 \times 3^3$ ways of getting seven predictions right out of ten. Hence, the number of ways of getting seven or more correct predictions is

$$\begin{aligned} {}^{10}C_{10} + {}^{10}C_9 \times 3 + {}^{10}C_8 \times 3^2 + {}^{10}C_7 \times 3^3 &= 1 + 30 + 405 + 3240 \\ &= 3676 \end{aligned}$$

The total number of predictions is 4^{10} since on each occasion there are four possible outcomes and one of them is right. Hence, the probability of the gambler getting seven or more correct predictions is

$$P = \frac{3676}{4^{10}} = \frac{3676}{1048576} = 0.00351 = 0.351\%$$