Picard's Method

Prerequisites

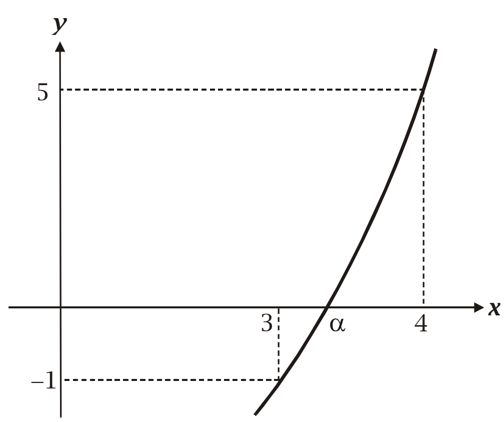
You should be familiar with the idea that a root of an equation is a value that makes that equation zero. If $y = f(x)$ is a function, then a *root* of the equation $f(x) = 0$ is a value $x = \alpha$ such that $f(\alpha) = 0$. You should also be familiar with the method of trial and improvement to find a root.

Example (1)
Find by trial and improvement the positive root of $y = x^2 - x - 7$ giving your answer to 2 d.p.

Given $y = x^2 - x - 7$ we first look for a sign change.

$y(0) = -7$
$y(1) = 1 - 1 - 7 = -7$
$y(2) = 4 - 2 - 7 = -5$
$y(3) = 9 - 3 - 7 = -1$
$y(4) = 16 - 4 - 7 = 5$

S0 $y < 0$ at $x = 3$ and $y > 0$ at $x = 4$, therefore the positive root lies between 3 and 4. A first approximation for the root is $\alpha = 3.5 \pm 0.5$, which is only accurate to 1 s.f.



We now look to improve the approximation by narrowing the interval. An obvious point to start is with $x = 3.5$.

$y(3.5) = (3.5)^2 - 3.5 - 7 = 1.75$	$y > 0$	$3 < \alpha < 3.5$
$y(3.2) = (3.2)^2 - 3.2 - 7 = 0.04$	$y > 0$	$3 < \alpha < 3.2$

$y(3.1) = (3.1)^2 - 3.1 - 7 = -0.49$ $y < 0$ $3.1 < \alpha < 3.2$
$y(3.15) = (3.15)^2 - 3.15 - 7 = -0.2275$ $y < 0$ $3.15 < \alpha < 3.20$
$y(3.18) = (3.18)^2 - 3.18 - 7 = -0.0676$ $y < 0$ $3.18 < \alpha < 3.20$
$y(3.19) = (3.19)^2 - 3.19 - 7 = -0.0139$ $y < 0$ $3.19 < \alpha < 3.20$
$y(3.195) = (3.195)^2 - 3.195 - 7 = 0.013025$ $y > 0$ $3.19 < \alpha < 3.195$

Therefore, $\alpha = 3.19$ (2 d.p.)

Trial and improvement is an example of an *iterative method*. The term "iterative" means that the same process is repeated again and again. In this case we repeat the process of trying an x value chosen from an interval which is known to contain the root. We repeat this until a numerical value for the root is obtained to the required degree of accuracy. Trial and improvement is a slow method of finding a root, meaning that many iterations (repeats) have to be made in order to arrive at a value. In this chapter we introduce another iterative method for finding a root. This is known as Picard's method.

Finding roots by Picard's method

Suppose α is a root of $g(x) = 0$. Then α can sometimes be found by rearranging $g(x)$ to obtain an expression $x = F(x)$. That is

$$g(x) = F(x) - x = 0$$

Then the iteration formula

$$x_{n+1} = F(x_n)$$

can be used to find the root α. The use of this rearrangement of an equation is called Picard's method. Picard's method is also called fixed-point iteration. Sometimes this formula does not work because the values get larger and larger (diverge), not smaller and smaller (converge). We require the values to converge. The formula $x_{n+1} = F(x_n)$ only converges if $|F'(x)| < 1$ near the solution α. We will illustrate the method and discuss the conditions for convergence at the same time.

Example (2)

Solve $g(x) = 0$ where $g(x) = x^3 + 3x - 1$

Solution

$$x^3 + 3x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{3}(1 - x^3)$$

Hence, Picard's method gives the iteration formula

$$x_{n+1} = \frac{1}{3}\left(1 - x_n^{\ 3}\right)$$

We need a starting value for the iteration. Now

$$g(0) = -1$$
$$g(1) = 3$$

So by linear interpolation $x_0 = 0.25$ should be a good starting value for the iteration. Then

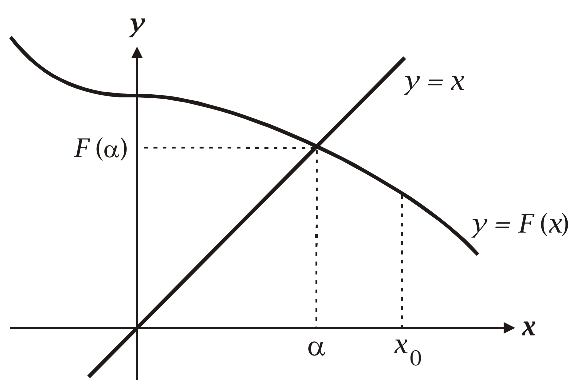
$$x_1 = \frac{1}{3}\left(1 - x_0^{\ 3}\right) = \frac{1}{3}\left(1 - (0.25)^3\right) = 0.328125...$$

Likewise

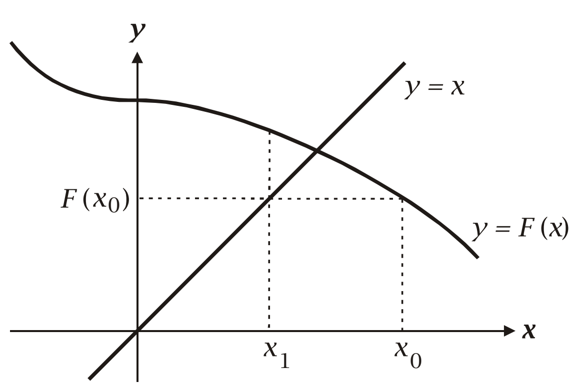
$$x_2 = 0.321557...$$
$$x_3 = 0.322250...$$

To 3 decimal places $x_2 = x_3 = 0.322$, hence, α = 0.322 (3 d.p.)

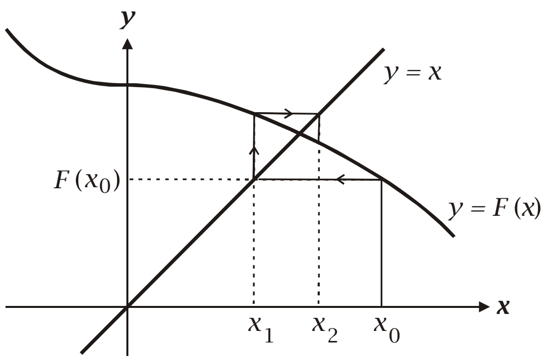
A graph illustrates why the method works, if it does.



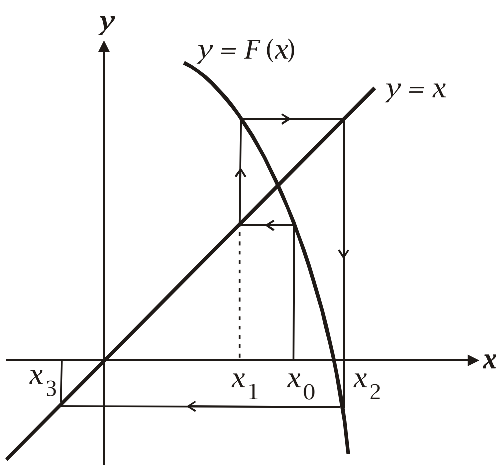
The diagram shows the graph of $y = F(x)$ and $y = x$ and the initial approximation, x_0. At the point of intersection, where $x = \alpha$, we have $F(\alpha) = \alpha$, hence $g(\alpha) = F(\alpha) - \alpha = 0$. So this is a solution of $g(x) = 0$. The value of the first approximation is shown in the following diagram.



At the point of intersection $y = F(x_0)$ with $y = x$ we have the value of the second approximation, x_1. The following diagram shows how repetition of this process causes successive approximations to converge on the root α.



However, when the gradient of $y = F(x)$ is greater than 1 around the root α, then the series of successive terms generated by the method diverges away from α, as the following diagram illustrates



So for the method to work we must have $|F'(x)| < 1$. Furthermore, the rate of convergence is faster, the smaller $|F'(x)|$ is.

Example (2)
Prove that the equation $2x = \ln\left(x + \sqrt{x^2 + 1}\right) + 1$ has a root in $[0,1]$. Find it correct to 6 decimal places.

Solution

The equation may be rearranged as

$$x = \frac{1}{2}\ln\left(x + \sqrt{x^2+1}\right) + \frac{1}{2}$$

which puts it in the form $x = f(x)$ required for Picard's method. Let

$$f(x) = \frac{1}{2}\ln\left(x + \sqrt{x^2+1}\right) + \frac{1}{2}, \quad g(x) = f(x) - x$$

Since $g(0) = f(0) - 0 = \frac{1}{2} > 0$ and $g(1) = f(1) - 1 = 0.94 - 1 = -0.06 < 0$, there is a root of the equation $g(x) = 0$ between 0 and 1, which is also a root of the equation

$$2x = \ln\left(x + \sqrt{x^2+1}\right) + 1 .$$

To confirm that the iteration will converge, we calculate the derivative of *f*, i.e. $f'(x) = \frac{1}{2} \times \frac{1}{\sqrt{1+x^2}}$. Therefore

$$\left|f'(x)\right| = \frac{1}{2\sqrt{1+x^2}} \le \frac{1}{2} < 1 \quad \text{for all } x \in [0,1].$$

So we can apply the iterative method. For this, let

$$x_0 = 0$$

$$x_{n+1} = \frac{1}{2}\ln\left(x_n + \sqrt{x_n^2+1}\right) + \frac{1}{2}, \quad \text{for all } n \ge 0$$

i.e. $x_{n+1} = f(x_n)$

$x_1 = f(x_0) = 0.5$
$x_2 = f(x_1) = 0.7406059$
$x_3 = f(x_2) = 0.8428074$
$x_4 = f(x_3) = 0.8828737$
$x_5 = f(x_4) = 0.8980413$
$x_6 = f(x_5) = 0.9037051$
$x_7 = f(x_6) = 0.9058092$
$x_8 = f(x_7) = 0.9065893$
$x_9 = f(x_8) = 0.9068783$
$x_{10} = f(x_9) = 0.90698542$
$x_{11} = f(x_{10}) = 0.9070250$
$x_{12} = f(x_{11}) = 0.9070397$
$x_{13} = f(x_{12}) = 0.9070451$
$x_{14} = f(x_{13}) = 0.9070472$
$x_{15} = f(x_{14}) = 0.9070479$

Therefore $\alpha = 0.907048$ (6 d.p.) since $g(0.9070475) > 0$ and $g(0.9070485) < 0$.