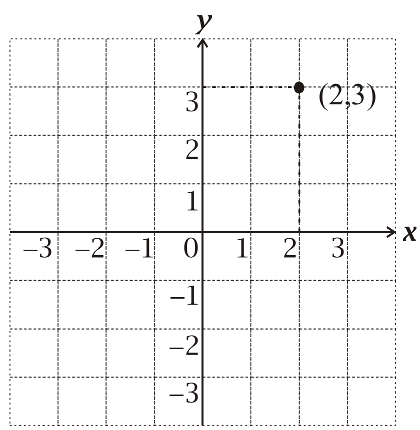
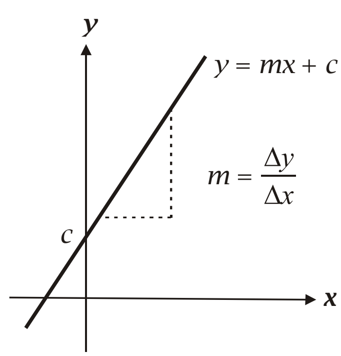
Polar Coordinates

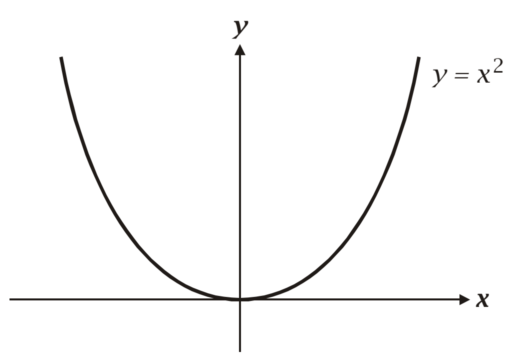
Cartesian coordinates revisited

Cartesian coordinates describe the position of a point on a two-dimensional flat surface by reference to a pair of axes which are perpendicular to one another. (Another term for perpendicular is *orthogonal.*) The position of a point is also specified by giving its coordinates relative to a fixed point called the origin. For example, the coordinates (2,3) correspond to a point thus

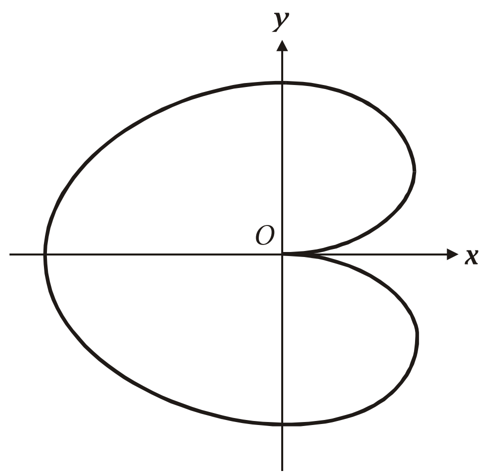


Cartesian coordinates are useful in the representation of such objects as straight lines and parabolas. These are shown as a relationship between the coordinates. The equation of the straight line is $y = mx + c$ where m is the gradient and c is the intercept. The equation of a parabola in Cartesian coordinates is $y = x^2$.





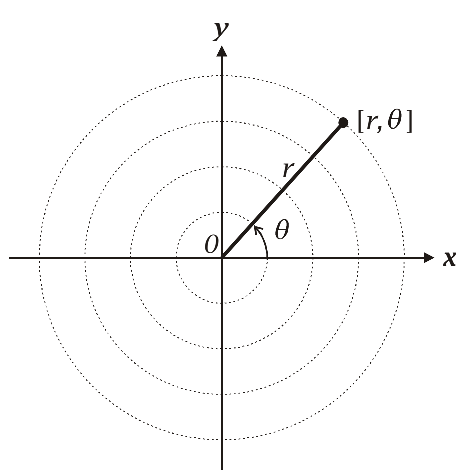
These simple examples illustrate the usefulness of Cartesian coordinates as a means of algebraically representing geometric objects. In other words, a coordinate system turns a geometric object into an algebraic one. Then algebraic and analytic techniques can be used to deduce other properties of these objects. But there are some geometric shapes that the Cartesian coordinate system is not so good at representing. For example, try representing in Cartesian coordinates this heart shape (called the *cardioid*).



The cardiod can be represented with greater ease using *polar coordinates.*

Polar coordinates

In the polar coordinate representation a point is identified by an angle and a distance from the origin. The angle is by convention the angle made by the line joining the point to the origin and the x–axis and is measured counter-clockwise.



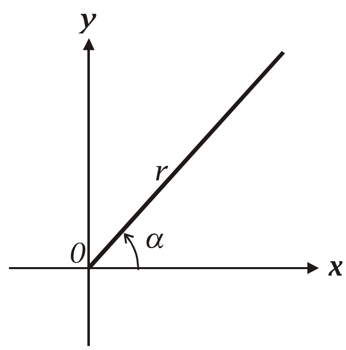
As the diagram indicates a point is represented in polar coordinates as an ordered pair of numbers $[r,\theta]$. We use the square brackets to indicate that these are polar coordinates and not Cartesian ones which use round brackets (x, y). By convention the distance from the origin comes first, and the angle second. Usually, the angle is given in radians unless otherwise stated.

The polar equation of the straight line and circle

The polar coordinate representation of a *half* line passing through the origin is given by the equation

$\theta = \alpha$

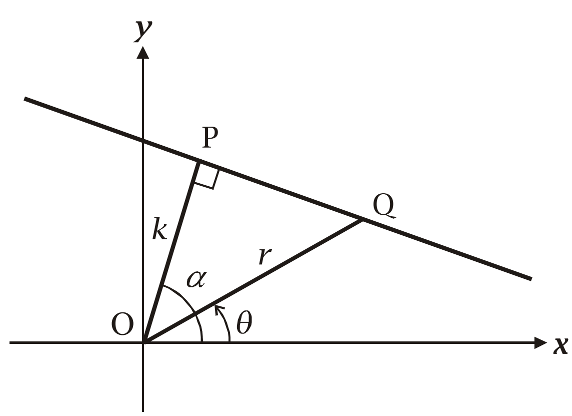
where α is a constant .



The equation of a straight line in general is given by

$k = r\cos(\alpha - \theta)$

where k and α are constants . In order to show this, suppose the perpendicular distance between the line and the origin is k.



The point Q represents a general point on the line with polar coordinates $[r, \theta]$. The line OP is a fixed line making a perpendicular with the line whose equation we seek. The distance OP is

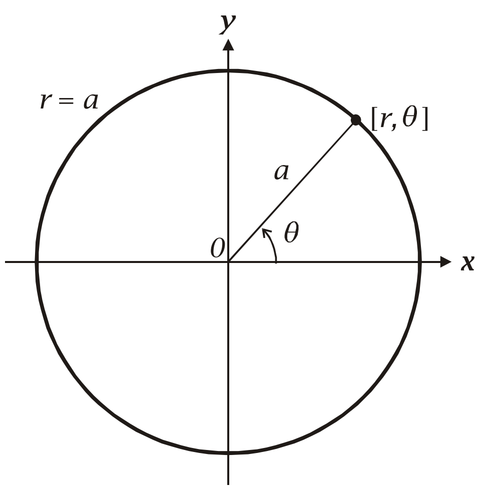
$OP = k$. OP also makes a fixed angle with the x -axis, α. The triangle OPQ is always a right-angled triangle in which

$$\cos(\alpha - \theta) = \frac{k}{r}$$

Rearrangement gives

$$k = r\cos(\alpha - \theta).$$

This shows that geometrical objects can be given a polar as well as a Cartesian representation. Since polar coordinates are circular coordinates, they are well adapted to the representation of objects that involve some form of circular motion. The simplest example is the circle whose equation is $r = a$ where a is a constant.



Curve sketching in polar coordinates

Curve sketching in polar coordinates is done by plotting points and joining them with smooth curves. This is best illustrated by example. Recall also that by convention angles are measured anti-clockwise starting from the positive x-axis.

Example (1)

Sketch the curve whose polar equation is

$$r = a(1 + \cos 3\theta)$$

Solution

To make this sketch we start by forming a table of values. Since we are using polar coordinates the arguments in the table will be angles, which are most suitably given in radian measure. The values are the corresponding distances, r, of the points on the curve from the origin.

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
r													

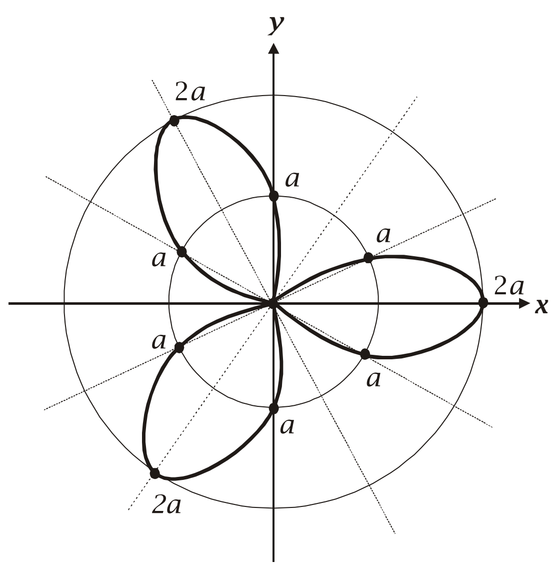
The values are found by substituting into the formula. Thus, for example when $\theta = \pi/6$.

$$\begin{aligned} r &= a(1+\cos 3\theta) \\ &= a\left(1+\cos\left(3\times \pi/6\right)\right) \\ &= a\left(1+\cos \pi/2\right) \\ &= a(1+0) \\ &= a \end{aligned}$$

By similar calculations the following table is obtained

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
r	$2a$	a	0	a	$2a$	a	0	a	$2a$	a	0	a	$2a$

There is no real alternative to this slow and usually tedious process of calculating each value. Finally, the curve is sketched by using radial coordinates to locate the points and joining them by smooth curves.



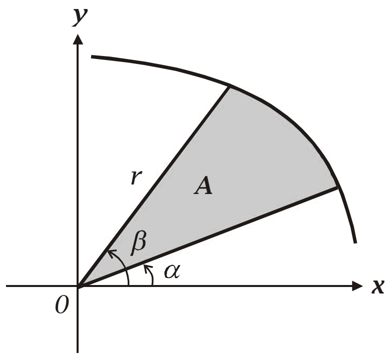
The equation defines a trefoil.

Areas of sectors in polar coordinates

The area of a sector in polar coordinates is given by

$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$

where α and β are the angles between which the area is to be found.



Before going on to prove this formula we illustrate its use.

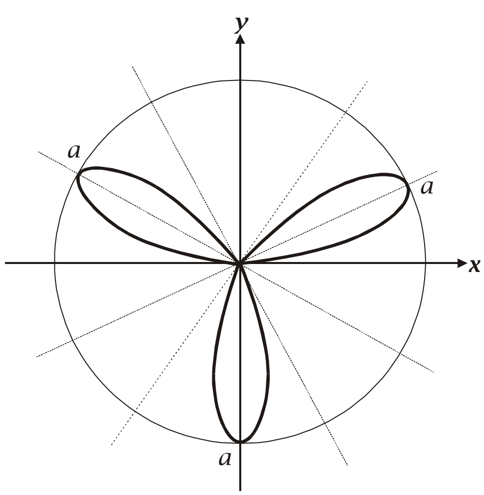
Example (2)

Sketch the curve $r = a\sin 3\theta$ and find the area of one loop.

Solution

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
r	0	a	0	$-a$	0	a	0	$-a$	0	a	0	$-a$	0

Negative values of r are not possible; consequently those regions where r takes a negative value simple do not represent a curve. Thus, the curve has sketch.



$$A = \int_0^{\pi/3} \frac{1}{2} r^2 d\theta$$
$$= \int_0^{\pi/3} \frac{1}{2} a^2 \sin^2 3\theta \; d\theta$$

This evaluation involves a trigonometric integration using formula

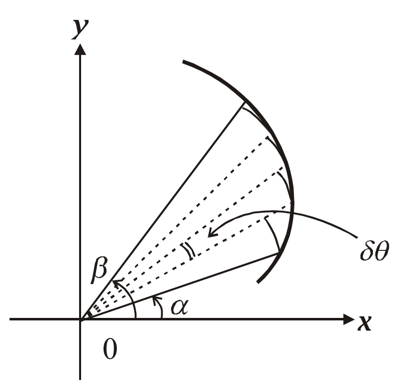
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$
$$\therefore \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$
$$\therefore \sin^2 3\theta = \frac{1}{2} - \frac{1}{2}\cos 6\theta$$

Hence

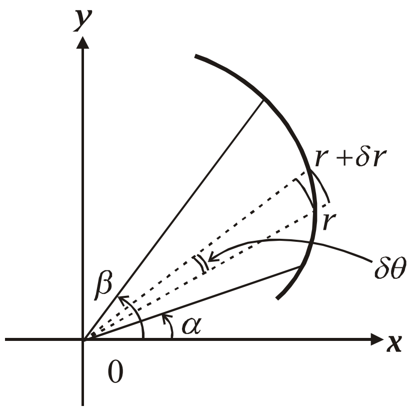
$$A = \int_0^{\pi/3} \frac{1}{2} a^2 \left(\frac{1 - 6\cos 6\theta}{2} \right) d\theta$$
$$= \frac{1}{4} a^2 \left[\theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/3}$$
$$= \frac{1}{4} a^2 \left(\frac{\pi}{3} \right)$$
$$= \frac{1}{12} a^2 \pi$$

Proof of the formula for the sector area in polar coordinates

We divide the area up into segments which are made of wedges centred on the origin.



Then the area is approximated by the total area of the wedges. Each wedge is made by sweeping out a small angle $\delta\theta$, and has area $\frac{1}{2} r^2 \delta\theta$.



Thus

$$A \approx \sum_{\theta=\alpha}^{\theta=\beta} r^2 \delta\theta$$

And in the limit as $\delta\theta \to 0$

$$A = \int_{\alpha}^{\beta} r^2 d\theta$$

LOOKS AS IF THERE IS AN ERROR IN THIS AND THE ½ HAS BEEN LEFT OUT.