Polynomial Algebra

Prerequisites

By this stage you should already be familiar with expressions of the form

$(x+2)(x^2-x-1)$ and x^3+x^2-3x-2

The purpose of this chapter is to summarise and consolidate the knowledge that you should already have acquired over some time regarding the manipulation of such expressions.

Polynomials

Expressions of the form

x^3+x^2-3x-2

are called *polynomials*. The expression x in this example is a *variable*. It stands for a number. Polynomials may have more than one variable. In the following there are two variables, x and y

$x^2y+2x^2+y^2+2y$.

Any letter or symbol can be used as a variable. Terms such as x^2y that form a part of such polynomials such as $x^2y+2x^2+y^2+2y$ are also called *algebraic terms*.

The manipulation of polynomials - algebra

There are a number of ways in which we can change a polynomial into another polynomial or expression. The manipulation of polynomials is called *algebra*. What follows is a summary of techniques used in algebra.

1. Substitution of values

A variable represents a quantity in general. At a given point in an algebraic process we may wish to *substitute* a particular *value* for a variable.

Example (1)

Substitute $x=2$, and $y=-1$ in $x^2y+2x^2+y^2+2y$.

Solution

When $x = 2$, and $y = -1$

$$x^2y + 2x^2 + y^2 + 2y = (2)^2 \times -1 + 2 \times (2)^2 + (-1)^2 + 2 \times -1$$
$$= -4 + 8 + 1 - 2$$
$$= 3$$

We also say that the expression $x^2y + 2x^2 + y^2 + 2y$ has been *evaluated* at $x = 2$, and $y = -1$.

2. Multiplication and division of algebraic terms

These follow the same rules as those for arithmetic. Recall that the multiplication of a positive by a negative number gives a negative number and that the multiplication of two negative numbers gives a positive number. Algebraic terms follow the same rules as the following examples illustrate.

(i) $x \times -x = -x^2$

(ii) $-2x \times -3y = 6xy$

(iii) $\dfrac{-5x}{2y} = -\dfrac{5x}{2y} = \dfrac{5x}{-2y}$

The order in which numbers are multiplied does not affect the value of the product. For instance, $4 \times 5 = 5 \times 4 = 20$. The same applies to algebraic terms. Thus, in algebra the term *pq* represents the same term as *qp*.

$pq = qp$.

3. Addition of algebraic terms

Consider the expression

$x^2y + 3x^2y + 3y - y$.

Here we observe that the first two terms in this expression have the same elements, excepting that the second is a multiple of the first.

First term	x^2y
Second term	$3x^2y$

These two terms have the same form – they are called *like terms*. The numbers in front of algebraic terms are called *coefficients*. The coefficient of the second term is 3; the coefficient of the first term is 1, for x^2y is the same as $1 \times x^2y$. Like terms can be added together

$x^2y + 3x^2y = 4x^2y$.

This process of adding together like terms is called *collecting* terms. In the expression

$x^2y+3x^2y+3y-y$

the last two terms are also *alike* and can be collected.

$3y-y=2y$.

Altogether

$x^2y+3x^2y+3y-y=4x^2y+2y$.

Note that x^2y and y are *not alike*, even though both contain a *y*. For terms to be alike they must have the same combination of variables., though these may be in any order In questions the instruction *simplify* requires you to collect like terms if any exist

Example (2)

Simplify $xy+2yx+zy-3yz$

Solution

$xy+2yx+zy-3yz=3xy-2yz$

Note how in this solution we are able to collect the terms in xy and yx because, although the order of terms is different in them, they are expressions with the same terms. The same applies to zy and $-2yz$. Note also that the solution could have been written as any of the following

$3xy-2yz$
$3yx-2yz$
$3xy-2zy$
$3yx-2zy$
$-2yz+3xy$
$-2yz+3yz$
$-2zy+3xy$
$-2zy+3yx$

These all stand for the same solution. Generally, we write solutions using letters in their usual alphabetic order, but this is for convenience only.

4. Cancellation of terms

In arithmetic we can cancel out common factors. For example

$$\frac{20}{15}=\frac{4\times5}{3\times5}=\frac{4\times\cancel{5}}{3\times\cancel{5}}=\frac{4}{3}.$$

We can perform the same operation of *cancelling down* with algebraic terms.

$$\frac{9p^2q}{3pq^2} = \frac{9 \times p \times p \times q}{3 \times p \times q \times q}$$
$$= \frac{\cancel{9}^3 \times \cancel{p} \times p \times \cancel{q}}{\cancel{3}^1 \times \cancel{p} \times q \times \cancel{q}}$$
$$= \frac{3p}{q}$$

5. Expanding brackets

When two algebraic terms are to be added first before being multiplied by another term, then we place those terms in a bracket. For example, in the expression

$$5(3x+2y) = 15x + 10y$$

The $3x$ is to be added to the $2y$ first and the whole is then multiplied by 5. This is different from

$$5 \times 3x + 2y = 15x + 2y$$

where the $3x$ only is multiplied by 5 and the $2y$ is not. Processes such as addition and multiplication are called *operations* and the brackets show the order in which the operations are to be performed (the *sequence of operations*). Thus

$$5(3x+2y)$$

means "first add $3x$ to $2y$, then multiply the whole lot by 5", whereas

$$5 \times 3x + 2y$$

means "first multiply $3x$ by 5 and then add this to $2y$". Observe also the rule (*convention*) in $5(3x+2y)$ that when a number is placed directly outside a bracket then this means that the number is to be multiplied by everything inside the bracket. When two collections of algebraic terms are placed in brackets side by side, then this means that all of the first bracket should be multiplied by all of the second.

$$(3x+2y)(x+y) = (3x+2y) \times (x+y)$$

Expanding brackets means to remove the brackets by multiplying all of the first by all of the second. To do this, the first term in the first bracket must be multiplied by each of the two terms in the second bracket, and likewise the second term in the first bracket must be multiplied by each of the two terms in the second bracket.

$$(3x+2y)(x+y) = 3x^2 + 3xy + 2xy + 2y^2.$$

Observe how as a result of this process we have the expression $3xy + 2xy$ in the result, which are like terms. These can be collected, so the whole operation of expanding the terms is

$$(3x+2y)(x+y) = 3x^2 + 3xy + 2xy + 2y^2$$
$$= 3x^2 + 5xy + 2y^2.$$

If you are required to do this in a question, then this is usually introduced with the instruction to *expand and simplify*.

Example (3)
Expand and simplify $(p+q)(p-q)$.

Solution

$$\begin{aligned}(p+q)(p-q) &= p^2+pq-pq-q^2\\ &= p^2-q^2\end{aligned}$$

In this example, the middle terms $pq-pq$ cancel out completely $pq-pq=0pq=0$. The process of expanding or removing brackets is also called *opening* the brackets.

6. Factorisation

In the previous example we expanded and simplified a pair of polynomials in brackets $(p+q)(p-q)=p^2-q^2$.

To go the other way and introduce brackets into a polynomial expression is called factorisation.

Example (4)
Factorise $ax+ay+bx+by$

Solution

$$\begin{aligned}ax+ay+bx+by &= a(x+y)+b(x+y)\\ &= (a+b)(x+y)\end{aligned}$$

In this example observe how in $a(x+y)+b(x+y)$ the term in the brackets, $x+y$, is repeated. They are like terms, so can be collected just like any other terms. Hence $a(x+y)+b(x+y)=(a+b)(x+y)$.

Factorisation is not as easy as expanding because it is not always obvious how a term should be factorised. For example in

$x^2+7x+10$

It is not immediately obvious whether this can be factorised and if so how. In fact, since

$$\begin{aligned}(x+2)(x+5) &= x^2+5x+2x+10\\ &= x^2+7x+10\end{aligned}$$

Then this can be reversed to give

$x^2 + 7x + 10 = (x+2)(x+5)$.

Although this process is mechanical and can be performed by a machine, for the student a certain amount of skill that can only be acquired through drill and practice is required to master to the process of factorisation. Later, certain techniques for finding factors are introduced. The process of finding factors is very important in mathematics.

7. Algebraic fractions

Algebraic fractions are multiplied, divided, added and subtracted according to the same rules that govern arithmetic. Consider the following examples.

(i) $$\frac{3a}{b^2} \times \frac{b}{9a^2} = \frac{3 \times a \times b}{b^2 \times 9 \times a^2} = \frac{1}{3ab}$$

(ii) $$\frac{6pq}{5(p+q)} \div \frac{2p^2}{10(p+q)} = \frac{6pq}{5(p+q)} \times \frac{10(p+q)}{2p^2}$$
$$= \frac{6q}{p}$$

(iii) $$\frac{x}{3} + \frac{x}{4} - \frac{x}{5} = \frac{20x + 15x - 12x}{60} = \frac{23x}{60}$$

Linear equations

An algebraic equation is an equation involving an unknown quantity

$6x - 11 = 25$.

Here x is an unknown quantity. An equation is called a *linear equation* when the variables are raised to the powers equal to 1 and are not multiplied together. Thus $6x - 11 = 25$ is a linear equation in x. Also, $6x + 3y = 25$ is a linear equation in x and y, but the following are *not* linear equations.

$6x^2 - 11 = 25$

$6xy - 11 = 25$

The first is not linear because it involves a power of x greater than 1; the second is not linear because in it two variables have been multiplied together.

Solving linear equations

To *solve* an equation means to find the value or values of the variables that make the equation true. Sometimes equations do not have solutions. However, linear equations in a single variable can be solved by algebraic manipulation, as the following examples show.

Example (5)

Solve the equation $6x-11=25$.

Solution

$$6x-11=25$$
$$6x=36$$
$$x=6$$

In this we follow the convention that the process of finding the solution is written on successive lines down the page. Each stage of algebraic manipulation follows from the preceding line. It is not usual to put absolutely every line of manipulation into a solution. For example, after the first line of the above solution we are really adding 11 to both sides of the equation

$$6x-11=25$$
$$6x-11+11=25+11$$
$$6x=36$$

By adding 11 to both sides of the equation we are able to cancel out the 11 on the left-hand side of the equation completely. Another way of looking at this is that we are *bringing* the 11 from the left-hand side of the equation to the right-hand side. As we do so we have to change its sign. Whenever you transfer (*bring*) any term across an equals sign $(=)$ the sign of the term is changed from plus $(+)$ to minus $(-)$ or vice-versa. Like factorisation the process of solving a linear equation is mechanical and can be performed by a machine, but for the student the skill of solving linear equations can only be acquired through drill and practice.

Example (6)

Solve the equation

$$\frac{x}{2}-\frac{(x-1)}{3}=2$$

Solution

$$\frac{x}{2}-\frac{(x-1)}{3}=2$$
$$\frac{3x-2(x-1)}{6}=2$$
$$3x-2(x-1)=12$$
$$x+2=12$$
$$x=10$$

The first step of this solution is to combine the fractions $\frac{x}{2}$ and $-\frac{(x-1)}{3}$ into a single fraction. To do so we follow exactly the same rule that we would use for fractions involving just numbers, which is to place the two fractions over their lowest common denominator

$$\frac{x}{2}-\frac{(x-1)}{3}=\frac{3x-2(x-1)}{6}=2.$$

At that point we can multiply both sides by 6 to *clear* the 6 off the bottom of the fraction on the left-hand side by multiplying both sides by 6.

$$\frac{3x-2(x-1)}{\cancel{6}}\times\frac{\cancel{6}}{1}=2\times 6$$

$$3x-2(x-1)=12$$

Some people jump from the first line directly to $3x-2(x-1)=12$. In their minds they are multiplying everything by the lowest common denominator, which here is 6, to clear (remove) all the fractions.

$$6\times\left(\frac{x}{2}\right)-6\times\left(\frac{(x-1)}{3}\right)=6\times 2$$

$$\cancel{6}^{3}\times\left(\frac{x}{\cancel{2}}\right)-\cancel{6}^{2}\times\left(\frac{(x-1)}{\cancel{3}}\right)=6\times 2$$

$$3x-2(x-1)=12$$

Both methods are equivalent and it is a matter of personal preference which one to adopt.