Polynomial Division

Prerequisites

You should be familiar with multiplying out brackets of algebraic terms.

Example (1)

Expand and collect $(x-1)(x^2+3x+2)+5$.

Solution

$$\begin{aligned}(x-1)(x^2+3x+2)+5 &= x^3+3x^2+2x-x^2-3x-2+5\\ &= x^3+2x^2-x+3\end{aligned}$$

You should also be familiar with long division.

Example (2)

(a) Use long division to find $56293 \div 13$. Show your working.

(b) What is the remainder when $56293 \div 13$?

Solution

(a)

$$\begin{array}{r} 4328 \\ 13\overline{)56273} \\ \underline{52} \\ 42 \\ \underline{39} \\ 37 \\ \underline{28} \\ 113 \\ \underline{104} \\ 9 \end{array}$$

(b) The remainder is 9.

There are terms to describe the various numbers that appear in a long division.

$$\begin{array}{rl} & \text{QUOTIENT} \\ \text{DIVISOR}\,\big) & \text{DIVIDEND} \\ & \vdots \\ & \text{-----------} \\ & \text{REMAINDER} \end{array}$$

In the last example the divisor is 13, the dividend is 56273, the quotient is 4328 and the remainder is 9. In the first example we saw that $(x-1)(x^2+3x+2)+5=x^3+2x^2-x+3$.

Example (3)

Given $(x-1)(x^2+3x+2)+5=x^3+2x^2-x+3$ state

(a) the divisor
(b) the dividend
(c) the quotient
(d) the remainder

when x^3+2x^2-x+3 is divided by $(x-1)$.

Solution

$(x-1)(x^2+3x+2)+5=x^3+2x^2-x+3$

may be rewritten as

$$
\begin{array}{r}
x^2+3x+2 \\
(x-1)\overline{)x^3+2x^2-x+3} \\
\\
\underline{\quad} \\
5
\end{array}
$$

whence

(a)	divisor	$x-1$
(b)	dividend	x^3+2x^2-x+3
(c)	quotient	x^2+3x+2
(d)	remainder	5

We now seek a technique whereby we can divide one polynomial by another.

Polynomial division

Polynomial division mimics the process of long division.

Example (4)

By means of polynomial division find

$(x^3+2x^2-x+3)\div(x-1)$

Solution

The first step is to write out the problem as if it were a problem in long division.

$$\begin{array}{r|l} & \ldots\ldots\ldots\ldots \\ (x-1) & x^3+2x^2-x+3 \end{array}$$

The problem is to find the quotient and remainder. Just as with long division, we take the divisor $(x-1)$ and ask how many times it goes into the leading term of the dividend (x^3+2x^2-x+3). The answer is x^2.

$$\begin{array}{r|l} & \qquad\qquad x^2 \\ (x-1) & x^3+2x^2-x+3 \\ & \underline{x^3-x^2} \\ & \qquad 3x^2-x \end{array}$$

As the diagram shows we continue to mimic long division by multiplying x^2 by the divisor $(x-1)$ and subtracting the product from the dividend. We then bring down the next term, and repeat the process.

$$\begin{array}{r|l} & \qquad\qquad x^2+3x \\ (x-1) & x^3+2x^2-x+3 \\ & \underline{x^3-x^2} \\ & \qquad 3x^2-x \\ & \qquad \underline{3x^2-3x} \\ & \qquad\qquad 2x+3 \end{array}$$

Here $x-1$ goes $3x$ times into $3x^2-x$, so we add $3x$ to the quotient and subtract the product of $3x$ and $(x-1)$ from the dividend. The remainder is $2x+3$ and the divisor may still go into this, so we repeat the process a final time.

$$\begin{array}{r|l} & \qquad\quad x^2+3x+2 \\ (x-1) & x^3+2x^2-x+3 \\ & \underline{x^3-x^2} \\ & \qquad 3x^2-x \\ & \qquad \underline{3x^2-3x} \\ & \qquad\qquad 2x+3 \\ & \qquad\qquad \underline{2x-2} \\ & \qquad\qquad\qquad 5 \end{array}$$

Here the remainder is 5, and as the divisor cannot be further divided, the division has come to an end. We have found that $x^3+2x^2-x+3=(x-1)(x^2+3x+2)+5$.

Example (5)

By means of polynomial division find the quotient and remainder when $(2x^3-x^2-x+18)\div(x+2)$. Is $x+2$ a factor of $2x^3-x^2-x+18$?

Solution

$$\begin{array}{r l}
 & 2x^2 - 5x + 9 \\
x + 2 \big) & 2x^3 - x^2 \quad - x \quad + 18 \\
 & \underline{2x^3 + 4x^2} \\
 & \qquad -5x^2 - x \\
 & \qquad \underline{-5x^2 - 10x} \\
 & \qquad\qquad 9x + 18 \\
 & \qquad\qquad \underline{9x + 18}
\end{array}$$

The quotient is $2x^2 - 5x + 9$ and the remainder is 0. Because the remainder is 0, $x + 2$ is a factor of $2x^3 - x^2 - x + 18$.

When there are missing terms in the dividend

If a term does not appear in the dividend, then we add it in with a zero coefficient. For example, if the dividend is $x^5 + 3x^3 - 2x + 1$ write this first as

$x^5 + 0x^4 + 3x^3 + 0x^2 - 2x + 1$

Note that the missing powers of x have been added back in with zero coefficients. This facilitates the working, as the next example shall show.

Example (6)

Divide $x^5 + 3x^3 - 2x + 1$ by $x - 2$.

Solution

First write $x^5 + 3x^3 - 2x + 1$ as $x^5 + 0x^4 + 3x^3 + 0x^2 - 2x + 1$

$$\begin{array}{r l}
 & x^4 + 2x^3 + 7x^2 + 14x + 26 \\
x - 2 \big) & x^5 + 0x^4 + 3x^3 + 0x^2 - 2x + 1 \\
 & \underline{x^5 - 2x^4} \\
 & \quad 2x^4 + 3x^3 \\
 & \quad \underline{2x^4 - 4x^3} \\
 & \qquad 7x^3 + 0x^2 \\
 & \qquad \underline{7x^3 - 14x^2} \\
 & \qquad\quad 14x^2 - 2x \\
 & \qquad\quad \underline{14x^2 - 28x} \\
 & \qquad\qquad 26x + 1 \\
 & \qquad\qquad \underline{26x - 52} \\
 & \qquad\qquad\qquad 53
\end{array}$$

$$x^5 + 3x^3 - 2x + 1 = (x - 2)\left(x^4 + 2x^3 + 7x^2 + 14x + 26\right) + 53$$

The general form of polynomial division

Notice that the general form of a polynomial division looks like

$$
\begin{array}{r|l}
 & g(x) \\
\hline
(x-\alpha) & f(x) \\
 & \vdots \\
\hline
 & R
\end{array}
$$

where $f(x)$ is the dividend, $(x-\alpha)$ is the divisor, $g(x)$ is the quotient, and R is the remainder. We can also write this as

$f(x) = (x-\alpha)g(x) + R$

which will be very useful when we come to consider the further topic under the heading of the *Remainder and Factor theorems*.