Proof of the Binomial Theorem

Binomial coefficients

The coefficients in the expansion of the expression $(a+b)^n$ are called binomial coefficients. They are denoted by

$$ {}^nC_r \text{ or } \binom{n}{r} $$

The symbol ${}^nC_r$ means "the *r*th coefficient of the expansion of $(a+b)^n$ provided we start with $r = 0$. For example, in

$$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4 .$$

we have,

$${}^4C_0 = 1, \qquad {}^4C_1 = 4, \qquad {}^4C_2 = 6, \qquad {}^4C_3 = 4, \qquad {}^4C_4 = 1.$$

A combination is simply a set or group of objects with no particular order chosen from a larger set or group. If we are choosing five people from a group of five, then there is clearly only one combination. But if we are choosing three people from a group of five, then there can be more than one combination.

The topic of combinations is dealt with in a separate unit, where permutations are also considered.

In general, the number of combinations of *r* objects chose from *n* different objects is denoted by

$${}^nC_r$$

If we are choosing *r* objects from *n*, then the number of ways of picking the first object is *r*; the number of ways of picking the second object is *r-1*; the number of ways of picking the third object is *r-2*, and so forth.

However, this gives the number of distinct permutations or orders, and in a combination we are not interested in the specific order of the elements. For example, the combination

A, B, C

is the same as the combination

B, A, C

So, in order to find all the combinations, we need to divide the total number permutations representing the same combination. In the above example of three elements, *A, B, C,* there are six permutations

A, B, C
A, C, B
B, A, C
B, C, A
C, A, B
C, B, A

That is, we divide by $3! = 6$

In general, if we are taking r elements from a set of n elements, we divide the number of permutations by $r!$

So,

$$^{n}C_{r} = \frac{n \times (n-1) \times (n-2) \times \ldots \times (n-(r+1))}{r!}$$

But it is easier to work this out using the formula

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

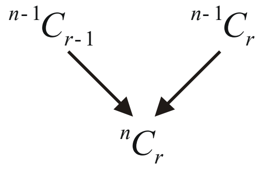
We will prove this formula; that is

$$
\begin{aligned}
{}^{n}C_{r} &= \frac{n\times(n-1)\times(n-2)\times\ldots\times(n-(r+1))}{r!} \\
&= \frac{n\times(n-1)\times(n-2)\times\ldots\times(n-(r+1))}{r!}\times\frac{(n-r)!}{(n-r)!} \\
&= \frac{n\times(n-1)\times(n-2)\times\ldots\times(n-(r+1))\times(n-r)!}{r!\ \times\ (n-r)!} \\
&= \frac{n!}{r!(n-r)!}
\end{aligned}
$$

We can write the rule for generating Pascal's triangle using these expressions for binomial coefficients as,

$$
{}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}
$$

That is,



$$
{}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}
$$

We will prove this result

$$
\begin{aligned}
{}^{n-1}C_{r-1} + {}^{n-1}C_{r} &= \frac{(n-1)!}{r!((n-1)-r)!} + \frac{(n-1)!}{(r-1)!((n-1)-(r-1))!} \\
&= \frac{(n-1)!}{r!((n-1)-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\
&= \frac{(n-r)(n-1)! + r(n-1)!}{r!(n-r)!}
\end{aligned}
$$

$$= \frac{(n-1)!(n-r+r)}{r!(n-r)!}$$

$$= \frac{n!}{r!(n-r)!}$$

$$= {}^nC_r$$

We will call this result, result (1) for the purpose of the proof of the Binomial Theorem, which follows

Another result that we will need is

$${}^kC_0 = {}^{k+1}C_0 = 1$$

and

$${}^kC_k = {}^{k+1}C_{k+1} = 1$$

We will call this result (2) and use it in the proof of the Binomial Theorem.

The Binomial Theorem

The use of binomial coefficients to expand binomial products is justified by the Binomial Theorem.

It is this that states that the expansion of $(a+b)^n$ is given by:

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + ... + {}^nC_1a^{n-r}b^r + ... + {}^nC_nb^n$$

$$= \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r.$$

The proof is by Mathematical Induction.

Proof

For $n = 1$

LHS $= (a+b)^1 = a+b = {}^1C_0a^1 + {}^1C_1b^1 =$ RHS

Therefore, the formula holds for $n = 1$

Induction step

Suppose the formula is true for $n = k$. That is,

$$\left(a+b\right)^k = \sum_{r=0}^{k} {}^kC_r a^{k-r} b^r$$

Then,

$$\begin{aligned}
\left(a+b\right)^{k+1} &= \left(a+b\right)\left(a+b\right)^k \\
&= \left(a+b\right) \times \sum_{r=0}^{k} {}^kC_r a^{k-r} b^r \qquad \text{by the Induction hypothesis} \\
&= \left(a+b\right)\left\{ {}^kC_0 a^k + {}^kC_1 a^{k-1} b + \ldots. + {}^kC_r a^{k-r} b^r + \ldots. + {}^kC_k b^k \right\} \\
&= {}^kC_0 a^{k+1} + {}^kC_1 a^k b + \ldots.. + {}^kC_r a^{k-(r+1)} b^{r+1} + \ldots.. + {}^kC_k a b^k \\
&\qquad + {}^kC_0 a^k b + {}^kC_1 a^{k-1} b^2 + \ldots.. + {}^kC_r a^{k-r} b^{r+1} + \ldots.. + {}^kC_k b^{k+1} \\
&= {}^kC_0 a^{k+1} + \left({}^kC_1 + {}^kC_0 \right) a^k b \ldots.. + \left({}^kC_r + {}^kC_{r-1} \right) a^{k-(r+1)} b^r + \ldots.. + \left({}^kC_k + {}^kC_{k-1} \right) a b^k + {}^kC_k b^{k+1} \\
&= {}^kC_0 a^{k+1} + {}^{k+1}C_1 a^k b \ldots.. + {}^{k+1}C_r a^{k-(r+1)} b^r + \ldots.. + {}^{k+1}C_k a b^k + {}^kC_k b^{k+1} \qquad \text{by result } (1) \\
&= {}^{k+1}C_0 a^{k+1} + {}^{k+1}C_1 a^k b \ldots.. + {}^{k+1}C_r a^{k-(r+1)} b^r + \ldots.. + {}^{k+1}C_k a b^k + {}^{k+1}C_{k+1} b^{k+1} \qquad \text{by result } (2) \\
&= \sum_{r=0}^{k+1} {}^{k+1}C_r a^{(k+1)-r} b^r
\end{aligned}$$

Therefore, the induction step holds, and the result is true for all *n*. That is, for all *n*

$$\begin{aligned}
\left(a+b\right)^n &= {}^nC_0 a^n + {}^nC_1 a^{n-1} b + \ldots. + {}^nC_1 a^{n-r} b^r + \ldots. + {}^nC_n b^n \\
&= \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r .
\end{aligned}$$