# Properties of Determinants

## Introduction

We first examine the properties of determinants of $3\times3$ matrices. Throughout this section we use the property

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

This relates the evaluation of a $3\times3$ determinant to the triple scalar product where **a**, **b** and **c** are the vectors, written in row form

$\mathbf{a} = (a_1, a_2, a_3)$ $\mathbf{b} = (b_1, b_2, b_3)$ $\mathbf{c} = (c_1, c_2, c_3)$

In this chapter we assume that these are vectors taken from the vector space $V \cong \mathbb{R}^3$, so the coordinates of the vectors are real numbers. $V = \mathbb{R}^3$ is a vector space over the field $\mathbb{R}$ [1]. We first develop statements and proofs of the basic properties of matrix determinants in this context. We state without proof that these properties generalise to determinants of square matrices of arbitrary dimension *n* over an arbitary field *K*. Proofs of the generalised version of these results are left for a subsequent chapter. For the present case where $\dim(V) = \dim(\mathbb{R}^3) = 3$ (That is $3\times3$ matrices) and where the field $K = \mathbb{R}$ we require the following properties.

Result, cylcic property of the triple scalar product

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \qquad (1)$$

Result

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \qquad (2)$$

## Properties of $3\times3$ determinants

**Property 1**

Interchanging rows multiplies the determinant by –1.

Proof

[1] Vectors have coordinates and these coordinates are *numbers* taken from a set of *numbers* having properties that make that set into a structure known as a *field*. In this context the field is just the set of real numbers. Fields of finite sets of numbers exist. The only fields of infinite sets of numbers are the real numbers, the complex numbers and quaternions. The expression $V \cong \mathbb{R}^3$ is read "*V* is isomorphic to $\mathbb{R}^3$", which means that it is essentially the same structure as $\mathbb{R}^3$, which is the usual 3 dimensional Euclidean space.

Proof is by exhaustion of cases. For example, suppose rows **b** and **c** are interchanged in

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Then we have

$$\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = \mathbf{a} \cdot (-\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\Delta$$

The other possibilities are from exchanging rows **a** and **b** and from exchanging rows **a** and **c**. The lead to the same result.

**Property 2**

If two rows of a determinant are identical, then the determinant is zero.

It is a property of the cross product that $\mathbf{a} \times \mathbf{a} = 0$ for any vector **a**.

Proof

If any two rows of a determinant are identical, then we can use the cyclic property, (1), above to permute determinant to obtain a form

$$\mathbf{b} \cdot (\mathbf{a} \times \mathbf{a}) = 0$$

Thus $\det \Delta = 0$.

**Property 3**

If a row of a determinant is multiplied by a scalar *k*, then the value of the resultant determinant is $k\Delta$.

Proof

To show this, for example, suppose row **a** is multiplied by *k*.

$$k\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = k\left(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\right) = k\Delta$$

**Property 4**

If one row from of a determinant is a linear combination of the other two rows then $\det \Delta = 0$.

Remark

For example, let row **a** be a linear combination of **b** and **c**; then

$$\mathbf{a} = k_1\mathbf{b} + k_2\mathbf{c}$$

Then

$$\begin{aligned}
\Delta &= \mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}) \\
&= (k_1\mathbf{b}+k_2\mathbf{c})\cdot(\mathbf{c}\times\mathbf{b}) \\
&= (k_1\mathbf{b})\cdot(\mathbf{c}\times\mathbf{b})+(k_2\mathbf{c})\cdot(\mathbf{c}\times\mathbf{b}) \\
&= k_1(\mathbf{b}\cdot(\mathbf{c}\times\mathbf{b}))+k_2(\mathbf{c}\cdot(\mathbf{c}\times\mathbf{b})) && [\text{Property 3}] \\
&= k_1(\mathbf{c}\cdot(\mathbf{b}\times\mathbf{b}))+k_2(\mathbf{b}\cdot(\mathbf{c}\times\mathbf{b})) && [\text{Cyclic permuation}] \\
&= 0 && [\text{Property of cross product}]
\end{aligned}$$

**Property 5**

If a row of a determinant is zero, then the determinant is zero.

Remark

This is a special case of property 4, where one row is zero times another, and hence a linear combination of it.

**Property 6**

If a linear combination of a set of rows of a determinant is added to another row not in the set, then the value of the determinant in not changed.

Proof

Proof is again by exhausting the possible cases. For example, suppose we add $k_1\mathbf{a}+k_2\mathbf{b}$ to the third row, $\mathbf{c}$. The new determinant is then

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\times(\mathbf{c}+k_1\mathbf{a}+k_2\mathbf{b})) &= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+\mathbf{a}\cdot(\mathbf{b}\times(k_1\mathbf{a}+k_2\mathbf{b})) \\
&= \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+0 && [\text{Property 4}] \\
&= \Delta
\end{aligned}$$

**Property 7**

$\det\mathbf{A}=\det\mathbf{A}^T$

Proof

$$\det\mathbf{A}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}+a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix}+a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)$$

$$\det \mathbf{A}^T = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) + b_1 (c_2 a_3 - a_2 c_3) + c_1 (a_2 b_3 - b_2 a_3)$$

Both of these expressions are the same; whence $\det \mathbf{A} = \det \mathbf{A}^T$.

Combining property 7 with the other properties, we can in effect exchange ‘row’ for ‘column’ in all the proofs.

# Generalisation

We state without proof that these properties can be generalised to determinants of any $3 \times 3$ matrix.[2]

# Evaluation of determinants

**Upper triangular form**

A $n \times n$ square matrix, **U**, is said to be in *upper triangular form* when all the entries in the matrix below the leading diagonal are zero.

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & 0 & u_{nn} \end{pmatrix}$$

<u>Theorem on upper triangular forms</u>

For a $n \times n$ square matrix, **U**, in upper triangular form the determinant is the product of the entries in the leading diagonal.

$$\det \mathbf{U} = \begin{vmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & 0 & u_{nn} \end{vmatrix} = u_{11} \times u_{12} \times \dots \times u_{nn}$$

**Technique for evaluating determinants**

[2] See the chapter, *Determinants of arbitary order*.

To evaluate the determinant of a square matrix, **A**

1. Use Gaussian row reduction to place the matrix into upper triangular form, **U**.
2. Use the above theorem to evaluate

$$\det \mathbf{U} = \begin{vmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & 0 & u_{nn} \end{vmatrix} = u_{11} \times u_{12} \times \dots \times u_{nn}$$

**Example**

Find the determinant of

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 2 & 3 & 6 \end{pmatrix}$$

Solution

Performing Gaussian row reduction on **A** to place it in upper triangular form

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 2 & 3 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \begin{matrix} (2) - (1) \\ (3) - 2 \times (1) \end{matrix}$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & \tfrac{3}{2} \end{pmatrix} \qquad (3) - \tfrac{1}{2} \times (2)$$

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & \tfrac{3}{2} \end{vmatrix} = 1 \times 2 \times \frac{3}{2} = 3$$