Properties of Matrix Multiplication

Prerequisites

You should be familiar with the addition and multiplication of matrices.

Operations

Addition and multiplication are examples of what mathematicians call *operations*. In mathematics an operation is any process that we do to one or more mathematical objects to obtain another object of that type. For example, we add two numbers to obtain another number – adding is an operation. The reason why we need the concept of an operation is that some of the things we do with and to numbers cannot be applied to other objects. So we need to be explicit about the properties of various operations.

We learn here that there are certain things that we take for granted with operations on numbers that when it comes to matrices either need to be proven, or are actually not true. Another way of putting this is that matrices are not numbers (they are arrays of numbers) and therefore sometimes behave like numbers and sometimes do not!

Associativity

An operation $*$ is said to be associative if, for any three entities, A, B, C doing first $A * B$ and then $(A * B) * C$ gives the same result as doing first $B * C$ and then $A * (B * C)$. It is the property

$$(A * B) * C = A * (B * C)$$

If the operation is associative then the bracket is irrelevant and we write

$$(A * B) * C = A * (B * C) = A * B * C$$

For example, for real numbers

$$(2 \times 3) \times 4 = 2 \times (3 \times 4) = 24$$

It does not matter how we bracket the individual numbers, so we can take the brackets off.

Example (1)

Let

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ -2 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}$$

By finding $(AB)C$ then $A(BC)$ show that $(AB)C = A(BC)$

Solution

$$\begin{aligned}(AB)C &= \left\{\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -2 & 3 \end{pmatrix}\right\} \begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 5 \\ 8 & -9 \end{pmatrix} \begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 26 & -17 \\ -50 & 33 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}A(BC) &= \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \left\{\begin{pmatrix} 0 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -4 & 3 \\ 2 & -1 \end{pmatrix}\right\} \\ &= \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 14 & -9 \end{pmatrix} \\ &= \begin{pmatrix} 26 & -17 \\ -50 & 33 \end{pmatrix}\end{aligned}$$

Hence

$(AB)C = A(BC)$

Example (2)

Assuming that multiplication of numbers is associative, prove that matrix multiplication of all square 2×2 matrices is associative.

Solution

Let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \qquad C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

We have to prove

(AB)C = A(BC)

where the operation is matrix multiplication. Then

$$
\begin{aligned}
(AB)C &= \left\{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right\}\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \\
&= \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \\
&= \begin{pmatrix} c_1\left(a_1b_1 + a_2b_3\right) + c_3\left(a_3b_1 + a_4b_3\right) & c_2\left(a_1b_1 + a_2b_3\right) + c_4\left(a_1b_2 + a_2b_4\right) \\ c_1\left(a_3b_1 + a_4b_3\right) + c_3\left(a_3b_2 + a_4b_4\right) & c_2\left(a_3b_1 + a_4b_3\right) + c_4\left(a_3b_2 + a_4b_4\right) \end{pmatrix} \\
&= \begin{pmatrix} c_1a_1b_1 + c_1a_2b_3 + c_3a_3b_1 + c_3a_4b_3 & c_2a_1b_1 + c_2a_2b_3 + c_4a_1b_2 + c_4a_2b_4 \\ c_1a_3b_1 + c_1a_4b_3 + c_3a_3b_2 + c_3a_4b_4 & c_2a_3b_1 + c_2a_4b_3 + c_4a_3b_2 + c_4a_4b_4 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A(BC) &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\left\{\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}\right\} \\
&= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1c_1 + b_2c_3 & b_1c_2 + b_2c_4 \\ b_3c_1 + b_4c_3 & b_3c_2 + b_4c_4 \end{pmatrix} \\
&= \begin{pmatrix} c_1a_1b_1 + c_1a_2b_3 + c_3a_3b_1 + c_3a_4b_3 & c_2a_1b_1 + c_2a_2b_3 + c_4a_1b_2 + c_4a_2b_4 \\ c_1a_3b_1 + c_1a_4b_3 + c_3a_3b_2 + c_3a_4b_4 & c_2a_3b_1 + c_2a_4b_3 + c_4a_3b_2 + c_4a_4b_4 \end{pmatrix} \\
&= (AB)C
\end{aligned}
$$

The mere fact that the proof is so tedious demonstrates the need to prove it! Thus, matrix multiplication is associative. This means that for matrices, just as for numbers, we do not need to put brackets around pairs of matrices. We can just write ABC to mean multiply the three matrices in any order you fancy!

Commutativity

An operation $*$, is commutative if

$$A * B = B * A$$

So the order in which the operation is performed does not matter. The multiplication of numbers is commutative. For example $2 \times 3 = 3 \times 2$, and we can multiply pairs numbers in any order we like. However, *matrix multiplication is not commutative*. If A and B are matrices then we cannot assume

$$AB = BA$$

Sometimes it happens to be true, but in general $AB \neq BA$. To show that $AB = BA$ is not generally true, we need a counter example.

Example (3)

Let

$$A = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ -2 & 3 \end{pmatrix}$$

Show that $AB \neq BA$

Solution

$$AB = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 8 & -9 \end{pmatrix} \qquad BA = \begin{pmatrix} 0 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -11 & -16 \end{pmatrix}$$

So $AB \neq BA$

You need to be very aware of the failure of commutatively for matrix multiplication. Multiplication of a matrix on the left is *not* equivalent to multiplication of a matrix on the right.

Distributivity

It is also possible to combine operations. An important relationship between two operations is *distributivity*, which is expressed by the formula

$$A(B+C) = AB + BC$$

The expression $A(B+C)$ says first add B to C and then multiply by A. The expression $AB + BC$ says first multiply A and B, then multiply B and C, and add the two results together. Distributivity means that the result is the same either way. We can take A inside the bracket $(B+C)$. This is a property that we unconsciously assume for numbers, and it is true of matrices as well, but has to be proven.

Example (4)

Prove the distributive law for square 2×2 matrices.

Solution

This question is left as an exercise for the reader. It is the first question linked to this chapter with a fully-worked solution.

Transpose of a matrix

For vectors it is assumed, for example, that

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$$

so we can swap between row and column vectors whenever the corresponding elements are equal. However, *as matrices* there are two different matrices. When a row is swapped for a column, or vice-versa, the result is called the *transpose* of the original matrix.

If a vector $\mathbf{x}$ is given as a column vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

then its transpose is the row vector

$$\mathbf{x}^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

Example (5)

Find the transpose of the column vector $\mathbf{x} = \begin{pmatrix} 13 \\ 19 \\ 8 \end{pmatrix}$

Solution

$$\mathbf{x}^T = \begin{pmatrix} 13 & 19 & 8 \end{pmatrix}$$

Henceforth, we assume that when a vector is specified, then it is given in column form, and its transpose is a row vector.

Example (6)

Find $\mathbf{x}^T A \mathbf{x}$ where $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

Solution

$$\begin{aligned} \mathbf{x}^T A \mathbf{x} &= \begin{pmatrix} 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \\ &= 26 \end{aligned}$$

The transpose of a 2×2 $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the matrix $\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$. The rows are written as columns and vice-versa.

Example (7)

Find the transposes of $A = \begin{pmatrix} 2 & -2 & 1 \\ -1 & 1 & 0 \\ 3 & -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 0 & -3 \end{pmatrix}$

Solution

$$A^T = \begin{pmatrix} 2 & -1 & 3 \\ -2 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \qquad B^T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 4 & -3 \end{pmatrix}$$

Example (8)

Prove that for all 2×2 matrices A, B

$(AB)^T = B^T A^T$

Solution

Let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

then

$$A^T = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \qquad B^T = \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}$$

$$\begin{aligned} AB &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} B^T A^T &= \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}\begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \\ &= \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix} \\ &= AB \end{aligned}$$