Proportionality

Linear Relationships

A variable represents a physical quantity that can take more than one value. For example, time and distance are variables.

Proportionality is concerned with particular relationships between variables. As one variable changes, so too can another. When we consider proportionality the way one variable changes with another can be represented in a graph as a straight line. We call such relationships *linear*. Proportionality is a linear relationship between two variables.

Generally, in these relationships we think of one variable as changing as a consequence of some change in the other variable. The variable that changes as a consequence of changes in the other variable is called the *dependent* variable. The variable that does not change, but rather causes the change, is called the *independent* variable.

For example, the speed at which a car is travelling (dependent variable) is related to the pressure on the throttle (independent variable).

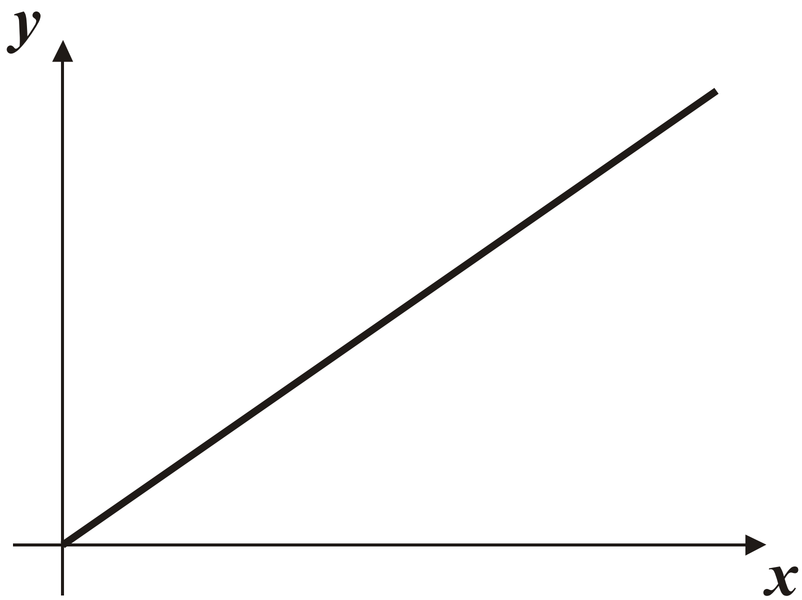
Proportionality is a linear relationship between two variables, where generally one is thought to depend on the other.

Actually, there are two kinds of linear relationship: *direct* and *indirect* proportionality.

Direct Proportionality

In direct proportionality, an increase in the independent variable causes an increase in the dependent variable.

This is represented graphically as a straight-line relationship with the dependent variable along the horizontal axis and the independent variable along the vertical axis.



The graph is a straight line through the origin. It is clear that the line representing this relationship has a constant gradient, *k*.

The symbol for proportionality is $\propto$. The expression,

$y \propto x$

means, "y is proportional to x".

The gradient is constant, hence

If $y \propto x$ then $y = k \times x$ where $k = \text{constant}$.

So far we have considered the possibility where y varies directly as x, but there is the possibility that y will vary as some power of x varies. For example, y could be proportional to the square of x. We represent this as $y \propto x^2$, and in general:

If $y \propto x^n$ then $y = k \times x^n$ where $k =$ constant.

Since k is the gradient of the line, it follows that any point on the line - that is any pair of values of x and y - is sufficient for us to calculate the value of the gradient.

<u>Example (1)</u>

The time taken (t) to mill barley is proportional to the mass (m) of barley to be milled. If it takes 4 hours to mill 100 kg of barley, express the relationship between the time taken and the mass of barely as an equation, and find the time that it will take to mill 1025 kg of barley.

Solution

We have, $t \propto m$

$\therefore \ t = k \times m$

$\therefore \ k = \frac{t}{m}$

When $m = 100,\ t = 4$

$\therefore \ k = \frac{4}{100} = 0.04$

$\therefore \ t = 0.04 \times m$

When $m = 1025,\ t = 0.04 \times 1025 = 41$ hours.

Relationships of proportionality can be combined. Thus,

If $y \propto x^n$ and $y \propto z^m$ then $y \propto x^n z^m$ and $y = k \times x^n \times z^m$ where k is a constant.

Example (2)

The energy (E) in a reservoir is proportional to the volume (V) of the water contained. The energy is also proportional to the height (h) of the reservoir above the electricity generating turbine. If the energy is 8×10^{11} joules when the volume is 2×10^6 m^3 and the height is 40 m, find the energy stored when the volume is 2.5×10^6 m^3 and the height is 50m.

Solution

$E \propto V$ and $E \propto h$

$\therefore E \propto h \times V$

$\therefore \ E = k \times h \times V$ where h is a constant

When $h = 40, V = 2\times10^{6},\ E = 8\times10^{11}$

$\therefore 8\times10^{11} = k\times40\times2\times10^{6}$

$$k = \frac{8\times10^{11}}{40\times2\times10^{6}} = 10000$$

$\therefore\ E = 10000\times h\times V$

When $h = 50$ and $V = 2.5\times10^{6}$, then

$E = 10000\times50\times2.5\times10^{6} = 1.25\times10^{12}$

Indirect proportionality

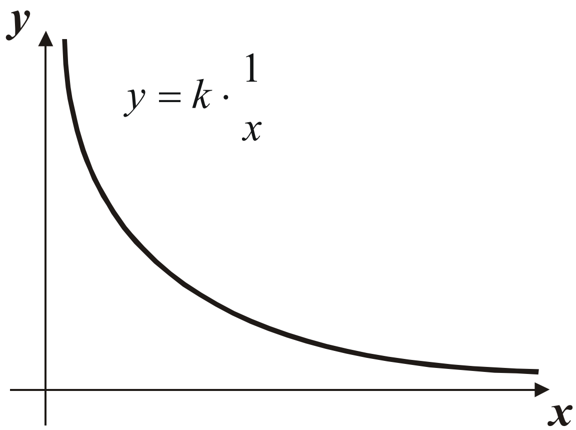
In indirect proportionality, as the independent variable goes up, the dependent variable goes down. We write this:

$$y \propto \frac{1}{x}$$

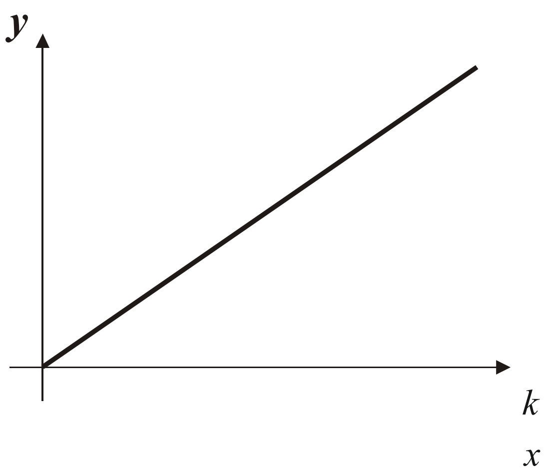
which means, "y is indirectly proportional to x". In this case, the constant of proportionality is introduced by:

$$y \propto \frac{1}{x} \text{ then } y = k\times\frac{1}{x}$$

The graph of y against x has the shape of a hyperbola. The curve is asymptotic to the x and y-axes - that is, the curve gets closer and closer to the x and y axes without actually touching.



The graph of y against $1/x$ shows the straight-line relationship of direct proportionality.



As with direct proportionality we can also have relations of indirect proportionality where

$$y \propto \frac{1}{x^n}$$

In this case a graph of y against $\frac{1}{x^n}$ would show a straight-line relationship.

Expressions involving relationships of inverse proportionality can be combined with each other, and with expressions involving relationships of direct proportionality.

If $x \propto \frac{1}{t^3}$ and $x \propto \frac{1}{A^2}$ and $x \propto S$, then

$x \propto \frac{S}{t^3 A^2}$ and

$x = \frac{kS}{t^3 A^2}$ where k is a constant.