Quadratic Inequalities

Prerequisites

You should be familiar both with linear inequalities and with quadratic polynomials. You should be able to find the factors of a quadratic of the from

$ax^2 + bx + c = 0$

and be able to sketch the corresponding quadratic polynomial

$y = ax^2 + bx + c$

by identifying its roots.

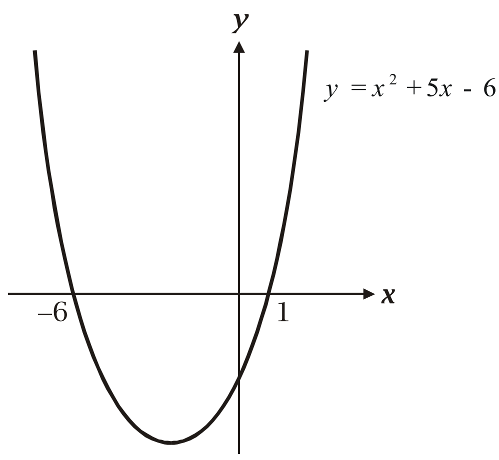
Example (1)

Find the roots of $y = x^2 + 5x - 6$ and sketch its graph.

Solution

$x^2 + 5x - 6 = 0$
$(x+6)(x-1) = 0$

The roots of $y = x^2 + 5x - 6$ are -1 and 6. Therefore $y = x^2 + 5x - 6$ crosses the x-axis at these points. It is also directed upwards since the coefficient of x^2 in $y = x^2 + 5x - 6$ is positive.



Solving quadratic inequalities

Quadratic inequalities are inequalities that involve an expression in x^2. For example, $x^2 + 5x - 6 > 0$ is a quadratic inequality. One way of solving such inequalities is by sketching the corresponding quadratic polynomial, and using the sketch to identify the range of values that satisfies the inequality.

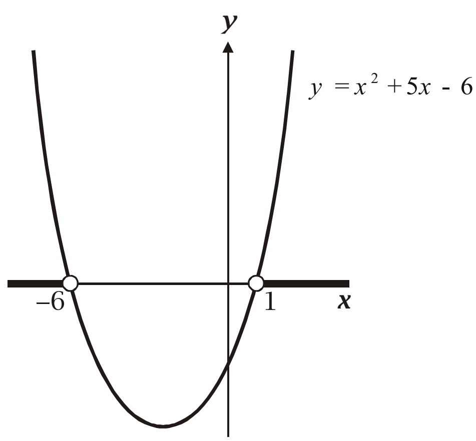
Example (2)

Solve the quadratic inequality $x^2 + 5x - 6 > 0$

Solution

$$x^2 + 5x - 6 > 0$$
$$(x+6)(x-1) > 0$$

The graph of $y = x^2 + 5x - 6$ (which we sketched in example 1) lies above the x-axis when $x < -6$ or $x > 1$.



Hence

$x^2 + 5x - 6 > 0$ when $x < -6$ or $x > 1$.

Dealing with fractions

Suppose you are required to solve the inequality $\frac{1}{x+1} < x$. The difficulty here is how to deal with the $x+1$ that comprises the denominator of the fraction $\frac{1}{x+1}$.

Example (3)

Explain why the following argument is *wrong*.

$$\frac{1}{x+1} < x$$
$$1 < x(x+1)$$

Solution

When $x > -1$

$x+1 > 0$

When $x < -1$

$x+1 < 0$

The rule for the manipulation of inequalities is that when multiplying both sides of an inequality by a negative number then the sign of the inequality must be reversed. But since $x+1$ can be both positive and negative we do not know whether to change the sign of the inequality or not.

There are two methods of dealing with fractions of this type.

<u>Method 1</u>

Since the square of any number is always positive, if we multiply both sides by the square of the denominator then we will remove the fraction. The sign of the inequality is unaffected.

Example (4)

Solve the inequality $\frac{1}{x+1} < x$ by this method

Solution

$$\frac{1}{x+1} < x$$
$$\frac{1}{(x+1)}(x+1)^2 < x(x+1)^2$$
$$(x+1) < x(x+1)^2$$
$$(x+1) - x(x+1)^2 < 0$$
$$(x+1)(1 - x(x+1)) < 0$$
$$(x+1)(1 - x^2 - x) < 0$$
$$(x+1)(x^2 + x - 1) > 0$$

At this point we need to factorise $x^2 + x - 1$. We use the quadratic formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to find its roots.

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 + 4}}{2}$$

$$x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$

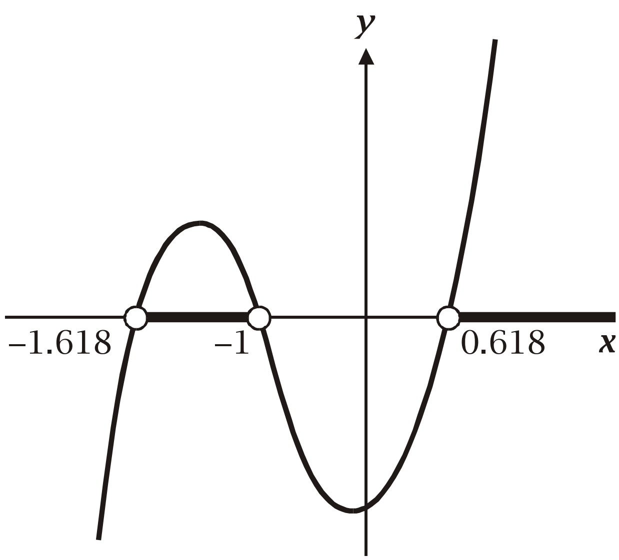
$x_1 = -1.618 \qquad x_2 = 0.618 \ (3.d.p.)$

Hence $\frac{1}{x+1} < x$ implies

$(x+1)(x+1.618)(x-0.618) > 0$.

To complete the solution we should sketch the graph of

$y = (x+1)(x+1.618)(x-0.618)$



Hence

$-1.618 < x < -1 \quad \text{or} \quad x > 0.618$

<u>Method 2</u>

Split the problem into two cases – the first where the fraction is positive and the second where the fraction is negative. Solve each case separately and combine the solutions at the end.

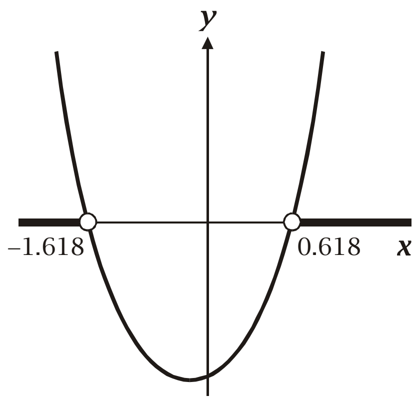
Example (4) continued

Solve $\frac{1}{x+1} < x$ by this method

Solution

If $x > -1$ we have $x + 1 > 0$. Therefore

$$\frac{1}{x+1} < x$$
$$1 < x(x+1)$$
$$1 < x^2 + x$$
$$x^2 + x - 1 > 0$$
$$(x+1.618)(x-0.618) > 0$$



$x > -1 \Rightarrow x < -1.618$ or $x > 0.618$

Now $x > -1 \Rightarrow x < -1.618$ is a contradiction, hence $x > 0.618$.

If $x < -1$ we have $x + 1 < 0$. Therefore

$$\frac{1}{x+1} < x$$
$$1 > x(x+1)$$

We must reverse the sign of the inequality because in this case $\frac{1}{x+1}$ is negative.

$$\frac{1}{x+1} < x$$
$$1 > x(x+1)$$
$$x^2 + x - 1 < 0$$
$$(x+1.618)(x-0.618) < 0$$

From the same graph above this implies

$x < -1 \Rightarrow -1.618 < x < 0.618$

The implies

$-1.618 < x < -1$

Combining the solutions from the two cases

$-1.618 < x < -1$ or $x > 0.618$.

This agrees with the solution by method 1.

In this case the first method is arguably slightly easier, and for problems at this level it is the most suitable method. However, in harder problems, where there are several fractions to clear, the second method becomes more efficient. In some difficult cases it may not be possible to use the first method.