Quadratic Polynomials

Prerequisites

A quadratic is an expression of the form $ax^2 + bx + c$ where *a*, *b* and *c* are numbers that we call *coefficients*. A *quadratic equation* is an expression $ax^2 + bx + c = 0$. To solve a quadratic means to find values of *x* that make the equation true. Such values are called *roots* of the equation. To find factors of a quadratic means to find values α and β such that

$$ax^2 + bx + c = (x - \alpha)(x - \beta)$$

One way of solving quadratic equations is by means of factorisation.

$$ax^2 + bx + c = 0$$
$$(x - \alpha)(x - \beta) = 0$$
$$x - \alpha = 0 \text{ or } x - \beta = 0$$
$$x = \alpha \text{ or } x = \beta$$

By this stage you should already be familiar with factorising quadratics. Let us revise and consolidate this.

Example (1)

Solve the equation

$$x^2 + 3x + 2 = 0$$

Solution

$$x^2 + 3x + 2 = 0$$
$$(x + 2)(x + 1) = 0$$

Since the left-hand side of this equation consists of two numbers that when multiplied together equal 0, then one of these numbers must be zero. Hence

$$(x + 2) = 0 \text{ or } (x + 1) = 0$$
$$x = -2 \text{ or } x = -1$$

Quadratic equations

However, it is not always possible to solve quadratic equations by direct factorisation.

Example (2)

Attempt to solve the quadratic equation

$3x^2 - 5x - 7 = 0$

by finding *whole* numbers α and β such that

$ax^2 + bx + c = (x - \alpha)(x - \beta)$

Describe the difficulties you encounter.

Solution

To factorise $3x^2 - 5x - 7$ we are looking for factors of 3 on the one hand with the factors of -7 on the other that can be combined in some way to give -5. There are just four possibilities

$$(3 \times 7) + (1 \times -1) = 20$$
$$(3 \times -7) + (1 \times 1) = -20$$
$$(3 \times 1) + (1 \times -7) = -4$$
$$(3 \times -1) + (1 \times 7) = 4$$

None of these yield the desired number -5, so there is no *whole number* solution to this problem.

Although there is no whole number solution to the problem that does not mean there are no solutions.

Example (3)

(*i*) Find the value of $3x^2 - 5x - 7$ when $x = \frac{5 + \sqrt{109}}{6}$.

(*ii*) Hence state one root of the equation $3x^2 - 5x - 7 = 0$.

Solution

When $x = \frac{5 + \sqrt{109}}{6}$

$$\begin{aligned} 3x^2 - 5x - 7 &= 3\left(\frac{5 + \sqrt{109}}{6}\right)^2 - 5\left(\frac{5 + \sqrt{109}}{6}\right) - 7 \\ &= \frac{1}{12}\left(25 + 10\sqrt{109} + 109\right) - \frac{25}{6} - \frac{5\sqrt{109}}{6} - 7 \\ &= \frac{1}{6}\left(67 + 5\sqrt{109} - 67 - 5\sqrt{109}\right) \\ &= 0 \end{aligned}$$

$x = \frac{5 + \sqrt{109}}{6}$ is a root of $3x^2 - 5x - 7 = 0$

This example demonstrates that the equation $3x^2 - 5x - 7 = 0$ can be solved. It turns out that there is a *quadratic formula* that will give the solution to quadratic equations, though we have to add the proviso *if they exist*, for not all quadratic equations do have solutions that are real numbers.

Quadratic formula

Firstly, we will present the quadratic formula and learn how to use it. Then we will demonstrate to you why it works. The formula is as follows.

If $ax^2 + bx + c = 0$

Then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The letters *a, b* and *c* stand for the numbers (*coefficients*) that appear in front of the terms x^2, x, and the constant. The use of the formula asks you to replace the symbols in the formula by these values from the equation. In the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

you will observe a "plus or minus" symbol $(\pm)$. This is a standard abbreviation for (in this context) for

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Example (3)

Use the quadratic formula to solve $3x^2 - 5x - 7 = 0$.

Solution

$3x^2 - 5x - 7 = 0$

The quadratic formula is

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

In this question

$a = 3,\ b = -5,\ c = -7$

Hence

$$x = \frac{5 \pm \sqrt{(-5)^2 - (4 \times 3 \times -7)}}{6}$$
$$= \frac{5 + \sqrt{109}}{6} \text{ or } \frac{5 - \sqrt{109}}{6}$$
$$= \frac{5 + 10.44...}{6} \text{ or } \frac{5 - 10.44...}{6}$$
$$= 2.57 \text{ or } -0.91 \text{ (2.D.P)}$$

In this solution we have given both the *exact* solution in terms of surds

$$x = \frac{5 + \sqrt{109}}{6} \text{ or } \frac{5 - \sqrt{109}}{6}$$

and a *numerical* and so *approximate* solution

$$x = 2.57 \text{ or } -0.91 \text{ (2.D.P)}$$

(We also state the degree of approximation; here to 2 decimal places.) Both solutions are correct and questions usually only expect you to offer only one of them. If one type of solution is preferred it will be stated in the question or clear from the context. The proof of quadratic formula follows from another technique that you must learn – the technique of completing the square – so firstly we describe that technique.

The technique of completing the square

Using this technique you can write any quadratic polynomial

$$y = ax^2 + bx + c$$

in the form

$$y = \mu(x - \alpha)^2 + \beta$$

where μ, α and β are numbers that we call *parameters*. We will show how we can derive these *parameters* from the original coefficients *a, b* and *c* in $y = ax^2 + bx + c$. Part of the importance of the form

$$y = \mu(x - \alpha)^2 + \beta$$

is that any quadratic polynomial that takes this form can be immediately sketched. We will also show how to do this. Firstly, let us turn our attention to the problem of deriving the parameters μ, α and β from the equation $y = ax^2 + bx + c$ and its coefficients *a*, *b* and *c*. This is learnt by example, and we will start with a simpler one.

Example (4)

(*i*) Expand $\left(x - \frac{5}{2}\right)^2$

(*ii*) Use the technique of completing the square to find numbers μ, α and β so that $y = x^2 - 5x - 11 = \mu\left(x - \alpha\right)^2 + \beta$.

Solution

(*i*)
$$\left(x - \frac{5}{2}\right)^2 = \left(x - \frac{5}{2}\right)\left(x - \frac{5}{2}\right)$$
$$= x - 5x + \left(\frac{5}{2}\right)^2$$
$$= x - 5x + \frac{25}{4}$$

(*ii*) As is so often the case in mathematics there is more than one way to solve this problem. We will describe two methods. The first employs direct algebraic manipulation to get $y = x^2 + 5x - 11$ into the required form. The second is a two-step process that involves equating coefficients. Both methods are equally valid and naturally arrive at the same solution. The first is quicker and the second is slightly easier to understand and use.

<u>Method 1</u>

Direct algebraic manipulation

$$y = x^2 - 5x - 11$$

The coefficient of the x term here is 5. Half of this number is $\frac{5}{2}$ and the square of this is $\left(\frac{5}{2}\right)^2$. Add *and* subtract $\left(\frac{5}{2}\right)^2$ simultaneously from the equation as follows.

$$y = x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 11$$

Note, firstly, because we have added and subtracted the same quantity we have really added 0, so the equation remains true. Secondly, note where we have placed the $\left(\frac{5}{2}\right)^2$, that is after the $5x$ term, and note also that we have not bothered to unsquare it. Although $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$ it is unnecessary to do this just now. The reason why it is unnecessary can be seen from the answer to part (*i*)

$$\left(x-\frac{5}{2}\right)^2 = x-5x+\left(\frac{5}{2}\right)^2$$

This enables us to write

$$y = x^2 - 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 11$$

directly as

$$y = \left(x-\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 11$$

It is because of this step that the technique is called "completing the square" because $\left(x-\frac{5}{2}\right)^2$ is a square. It remains only to "tidy up the backend of the equation".

$$\begin{aligned} y &= \left(x-\frac{5}{2}\right)^2 - \frac{25}{4} - \frac{44}{4} \\ &= \left(x-\frac{5}{2}\right)^2 - \frac{69}{4} \end{aligned}$$

Because of the annotation here needed to explain the technique, this method may appear long-winded. However, the complete solution, without annotation, takes at most four lines.

$$\begin{aligned} y &= x^2 - 5x - 11 \\ &= x^2 - 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 11 \\ &= \left(x-\frac{5}{2}\right)^2 - \frac{25}{4} - \frac{44}{4} \\ &= \left(x-\frac{5}{2}\right)^2 - \frac{69}{4} \end{aligned}$$

So, as we promised, it is fast. Now we will look at the second method.

Method 2

We have

$$y = x^2 - 5x - 11 = \mu\left(x-\alpha\right)^2 + \beta$$

We know already in this question that $\mu = 1$ because the coefficient of x^2 is 1. This simplifies the working.

$$y = x^2 - 5x - 11 = \left(x-\alpha\right)^2 + \beta$$

Expand the right-hand side of this

$$x^2 - 5x - 11 = x - 2\alpha + \alpha^2 + \beta$$

and equate coefficients

$$2\alpha = 5$$
$$\alpha^2 + \beta = -11$$

Solve these two equations

$$\alpha = \frac{5}{2}$$
$$\beta = 11 - \alpha^2 = -11 - \frac{25}{4} = -\left(\frac{44+25}{4}\right) = -\frac{69}{4}$$

$$y = \left(x - \frac{5}{2}\right)^2 - \frac{69}{4}$$

The process is only slightly longer than the first method, and is arguably easier to understand and remember.

This example was an "easier" one because the coefficient of x^2 in $y = x^2 + 5x - 11$ was $a = 1$. In the second method above this also entitled us to argue that $\mu = 1$ in $\mu(x-\alpha)^2 + \beta$. This simplified the working, but we should also look at a harder example.

Example (5)

Find the completed square form of $y = -2x^2 + 3x + 7$

Solution

Method 1

$$\begin{aligned} y &= -2x^2 + 3x + 7 \\ &= -2\left(x^2 - \tfrac{3}{2}x\right) + 7 \\ &= -2\left(x^2 - \tfrac{3}{2}x + \left(\tfrac{3}{4}\right)^2 - \left(\tfrac{3}{4}\right)^2\right) + 7 \\ &= -2\left(x - \tfrac{3}{4}\right)^2 + \frac{9}{8} + 7 \\ &= -2\left(x - \tfrac{3}{4}\right)^2 + 8\frac{1}{8} \end{aligned}$$

Annotated solution

$$y = -2x^2 + 3x + 7$$

There is a coefficient in front of the x^2 so place the x^2 term and the x term inside a bracket, adjusting the coefficient of the x term accordingly.

$$y = -2\left(x^2 - \tfrac{3}{2}x\right) + 7$$

Inside the bracket add and subtract half the square of the coefficient of the x term.

$$y = -2\left(x^2 - \tfrac{3}{2}x + \left(\tfrac{3}{4}\right)^2 - \left(\tfrac{3}{4}\right)^2\right) + 7$$

Remove the backend of the expression from the bracket, taking care to multiply it by the coefficient in front of the bracket and watching for any negative signs.

$$y = -2\left(x^2 - \tfrac{3}{2}x + \left(\tfrac{3}{4}\right)^2\right) - \left(-2 \times \left(\tfrac{3}{4}\right)^2\right) + 7$$

Inside the bracket complete the square and tidy-up the back-end.

$$y = -2\left(x - \tfrac{3}{4}\right)^2 + 8\frac{1}{8}$$

<u>Method 2</u>

Let

$$\begin{aligned} -2x^2 + 3x + 7 &= \mu(x - \alpha)^2 + \beta \\ &= \mu\left(x^2 - 2\alpha + \alpha^2\right) + \beta \\ &= \mu x^2 - 2\alpha\mu + \mu\alpha^2 + \beta \end{aligned}$$

Equating coefficients

$$\mu = -2$$
$$-2\alpha\mu = 3$$
$$\mu\alpha^2 + \beta = 7$$

and solving

$$\mu = -2$$
$$\alpha = \frac{-3}{2\mu} = \frac{3}{4}$$
$$\mu\alpha^2 + \beta = 7$$
$$-2\left(\frac{3}{4}\right)^2 + \beta = 7$$
$$\frac{9}{8} + \beta = 7$$
$$\beta = 7 + \frac{9}{8} = 8\frac{1}{8}$$

Therefore

$$y = -2\left(x - \tfrac{3}{4}\right)^2 + 8\frac{1}{8}$$

Proof of the Quadratic Formula

The proof of the quadratic formula proceeds by completing the square. To demonstrate the formula all we have to do is follow through the steps of the previous example but with the parameters *a*, *b* and *c* in place of their values.

Let $ax^2 + bx + c = 0$

Then $a\left(x^2 + \frac{b}{a}x\right) + c = 0$

$$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right) = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What this shows is that a quadratic equation can be solved directly from the completed square form without the use of the formula. This is because the formula really is just a short-cut method of completing the square.

Example (6)

In the preceding example we demonstrated that the completed square form of

$y = -2x^2 + 3x + 7$ is $y = -2\left(x - 3/4\right)^2 + 8\frac{1}{8}$.

Use this information to solve the equation $-2x^2 + 3x + 7 = 0$.

Solution

$$-2x^2 + 3x + 7 = 0$$

$$-2\left(x - 3/4\right)^2 + 8\frac{1}{8} = 0$$

$$-2\left(x - 3/4\right)^2 = -\frac{65}{8}$$

$$\left(x - 3/4\right)^2 = \frac{65}{16}$$

$$x - \frac{3}{4} = \pm\frac{\sqrt{65}}{4}$$

$$x = \frac{3}{4} \pm \frac{\sqrt{65}}{4}$$

Sketching quadratics

The completed square form of a quadratic $ax^2 + bx + c = 0$ is

$y = \mu(x - \alpha)^2 + \beta$

If a quadratic is in the completed square form then it is immediately possible to sketch its graph. The parameters μ, α and β provide all the information required for this purpose. To see why, let us first look in general at the graphs of quadratics. The graphs of *all* quadratics have the same basic shape, which is called a *parabola*.

Example (7)

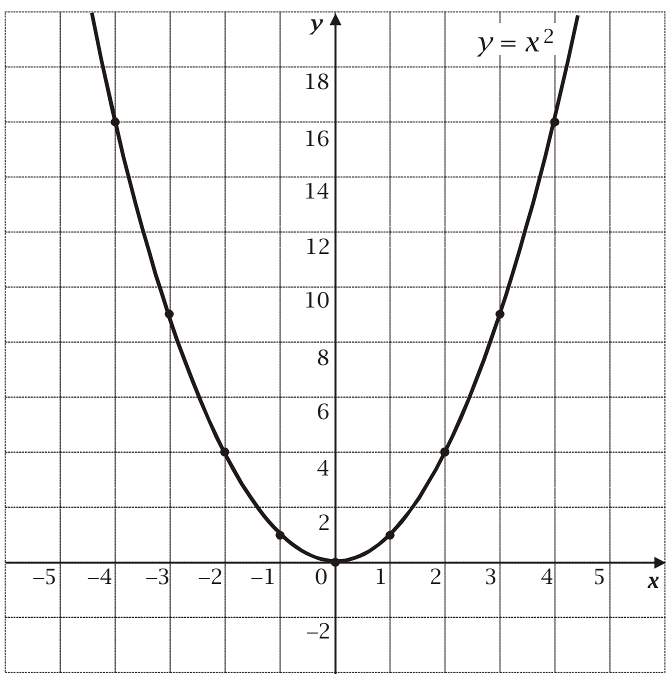
Complete the following table for the quadratic $y = x^2$.

x	–4	–3	–2	–1	0	1	2	3	4
y	16					1			

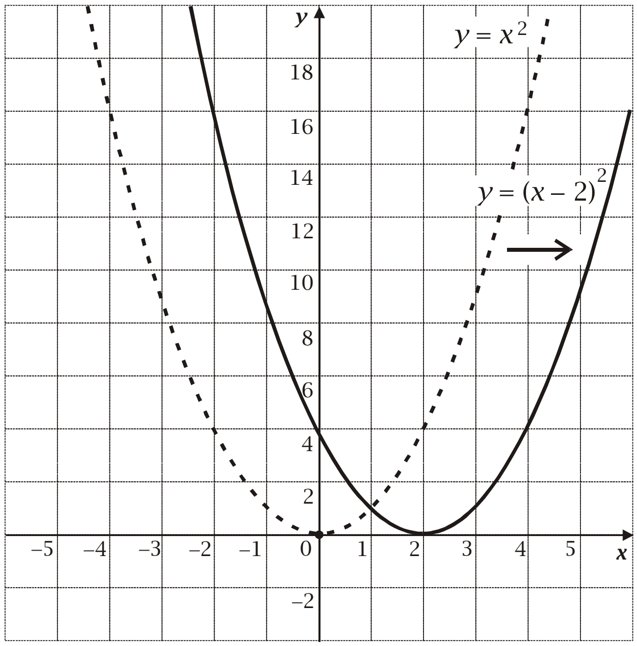
Use your results to plot the graph of $y = x^2$

Solution

x	–4	–3	–2	–1	0	1	2	3	4
y	16	8	4	1	0	1	4	8	16

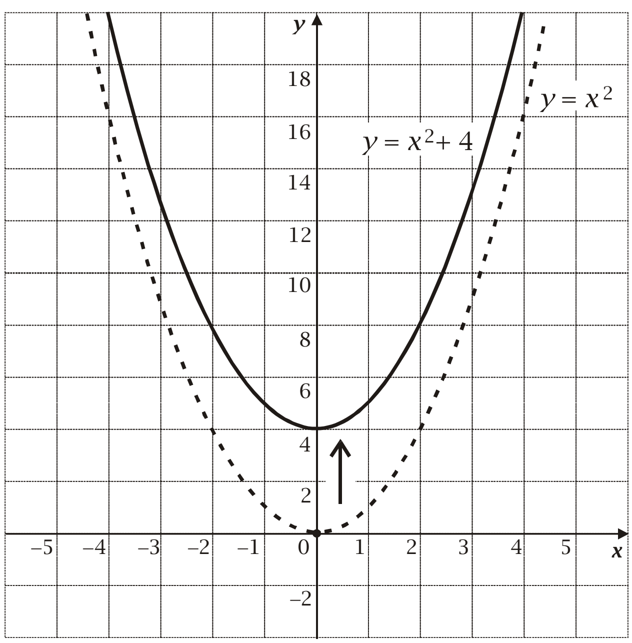


It is the shape of this graph that is called a *parabola*. Now observe that there are only a limited number of things that can be "done" to this graph that will change its position, orientation or aspect on the "page" without altering its basic shape. Firstly, it can be moved from side to side.



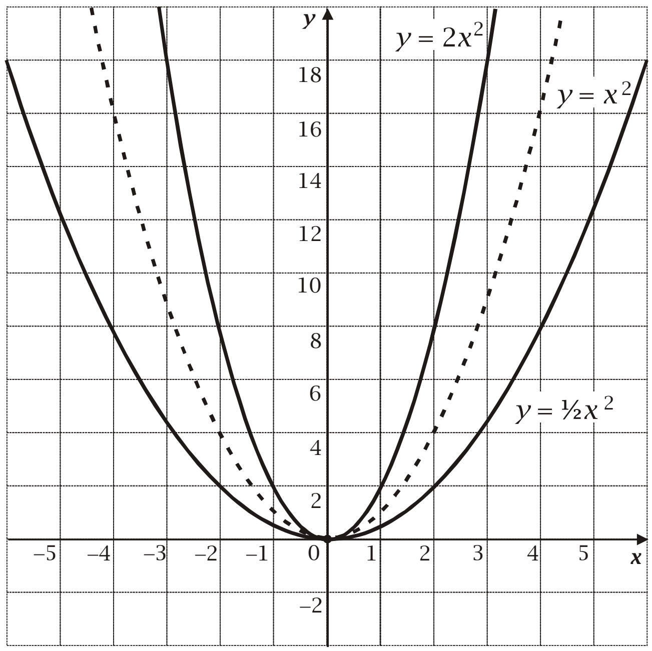
The graph of $y = (x - 2)^2$ *has the same shape as* $y = x^2$ *but has been shifted 2 units to the right.*

We can also move the graph up and down.



The graph of $y = x^2 + 4$ *has the same shape as* $y = x^2$ *but has been shifted up 4 units.*

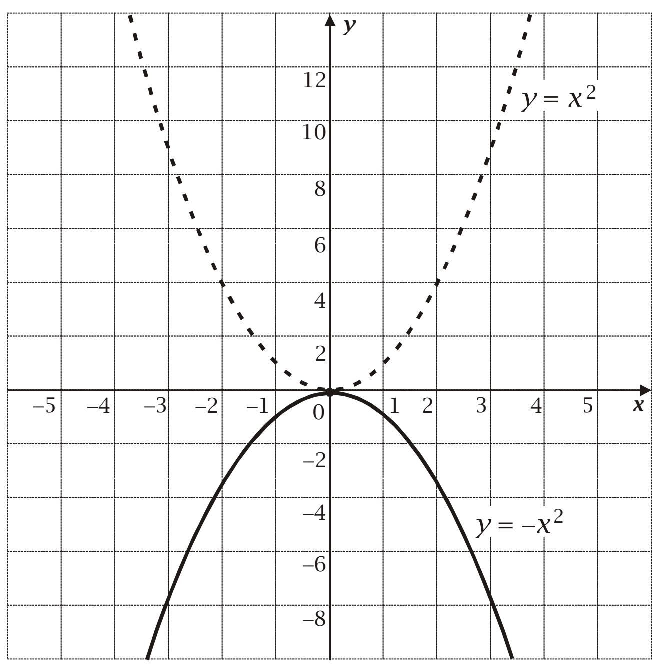
We can alter the steepness or shallowness of the parabola.



The graph of $y = 2x^2$ has the same shape as $y = x^2$ but is steeper.

The graph of $y = \frac{1}{2}x^2$ has the same shape as $y = x^2$ but is shallower.

We can also reflect the graph in the *x*-axis

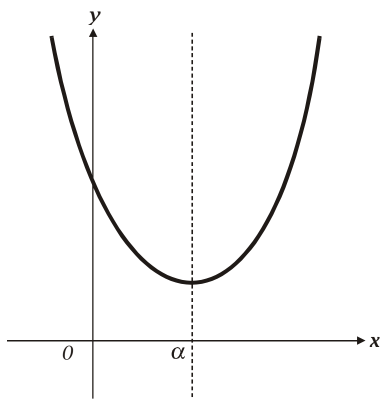


The graph of $y = -x^2$ is the reflection of $y = x^2$ in the x-axis.

These processes of horizontally and vertically shifting the parabola, of making it steeper or shallower or of reflecting it in the *x*-axis (or another line) are all called *transformations*. Clearly, transformations can also be combined, so that, for example, we can shift a graph horizontally and also make it steeper. However, the three parameters in the completed square form $y = \mu(x-\alpha)^2 + \beta$ are all that is required to describe every transformation of the parabola $y = x^2$. To see why, observe

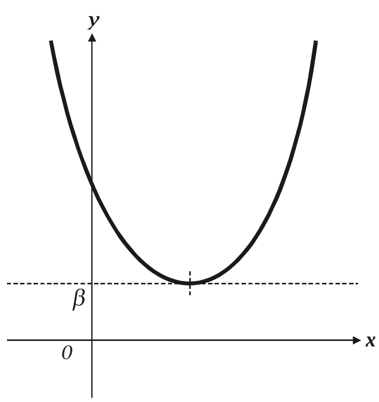
(1) The parameter α

This parameter gives the vertical *axis of symmetry* of the parabola $y = \mu(x-\alpha)^2 + \beta$.



(**2**) The parameter β

This parameter gives the minimum (or maximum) value of the parabola $y = \mu(x-\alpha)^2 + \beta$

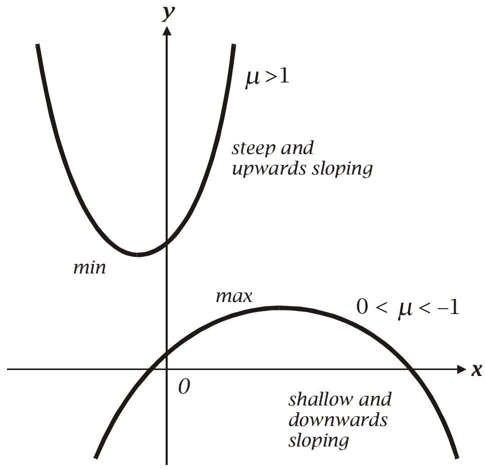


The minimum (or maximum) is located at (α, β)

(**3**) The parameter μ

This parameter describes the *scaling* of the parabola $y = \mu(x-\alpha)^2 + \beta$. If $\mu > 1$ then the graph is steeper than $y = x^2$. If $\mu < 1$ then the graph is shallower than $y = x^2$. If $\mu > 0$

(positive) then the graph is orientated upwards and has a minimum value, and if $\mu < 0$ (negative) then the graph is directed downwards and has a maximum value.



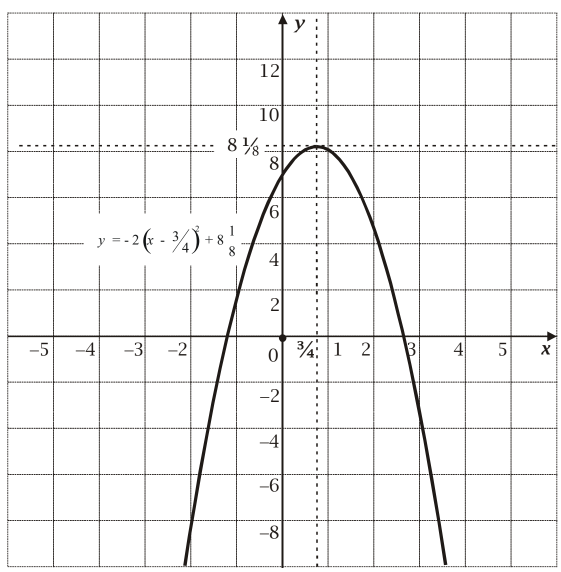
Examples of graphs showing the effect of changes in the value of the parameter μ on the aspect of a parabola

Example (8)

Sketch the graph of $y = -2\left(x - \frac{3}{4}\right)^2 + 8\frac{1}{8}$.

Solution

The axis of symmetry is $x = \frac{3}{4}$. The maximum point is at $\left(\frac{3}{4}, 8\frac{1}{8}\right)$. It is steeper than $y = x^2$ and downwards sloping. So the sketch is



Discriminant

We have seen that the solution to the equation $ax^2 + bx + c = 0$ is given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this formula there is an expression

$$\Delta = b^2 - 4ac$$

inside the square root symbol. If this expression is less than zero then the square root does not have solutions. This expression is called the discriminant.

It has already been mentioned that not all quadratics have real number solutions (*real roots*). The curve sketching that we have just learnt enables us easily to see why. The roots of a quadratic equation $ax^2 + bx + c = 0$ correspond to those values of x where the graph crosses the x-axis.

Example (9)

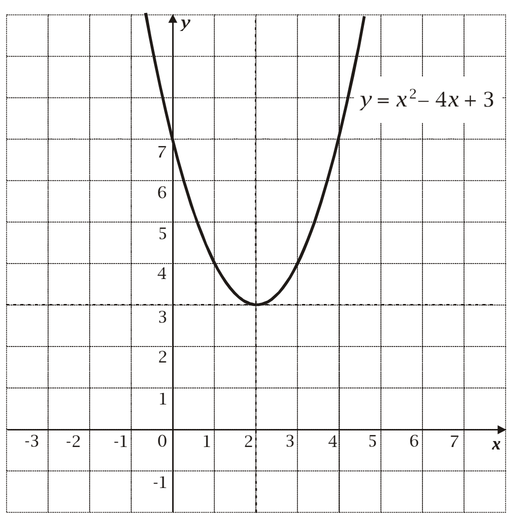
(*i*) Find the completed square form of $y = x^2 - 4x + 7$ and sketch its graph.

(*ii*) Use the graph sketched in part (*i*) to state how many solutions there are to the equation $x^2 - 4x + 7 = 0$.

Solution

(*i*)

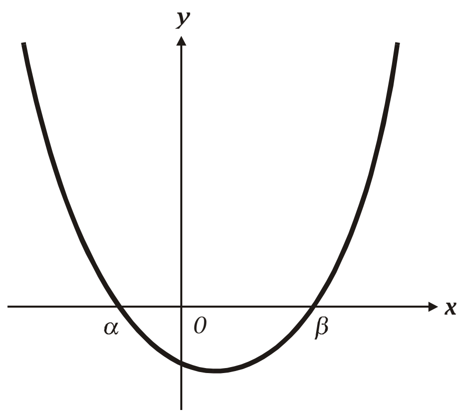
$$\begin{aligned} y &= x^2 - 4x + 7 \\ &= x^2 - 4x + (2)^2 - (2)^2 + 7 \\ &= (x - 2)^2 + 3 \end{aligned}$$



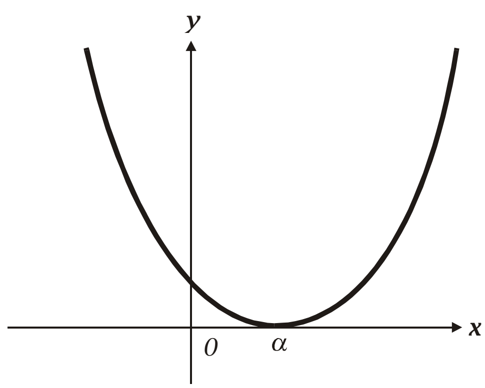
(*i*) The graph shows that there are no (zero) solutions.

There is no need to actually complete the square of a quadratic polynomial in order to determine how many solutions it has. The discriminant is all that is needed. Recall that the discriminant is the number $\Delta = b^2 - 4ac$.

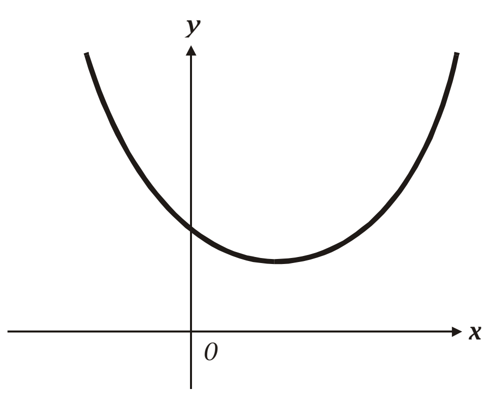
(1) If $\Delta > 0$ then the quadratic polynomial has two real roots.



(2) If $\Delta = 0$ then the quadratic polynomial has a single real root.



(3) If $\Delta < 0$ then the quadratic polynomial does not have real roots.



Example (10)

For each of the following quadratic equations, determine the number of real roots.

(a) $x^2 + 2x - 3 = 0$

(b) $x^2 - 6x + 9 = 0$

(c) $x^2 - 2x + 4 = 0$

Solution

(a) $y = x^2 + 2x - 3$

$\Delta = 4 + 12 = 16 > 0$

Therefore $x^2 + 2x - 3 = 0$ has 2 real solutions.

(b) $y = x^2 - 6x + 9$

$\Delta = 36 - 36 = 0$

Therefore $x^2 - 6x + 9 = 0$ has 1 real solution.

(c) $y = x^2 - 2x + 4$

$\Delta = 4 - 16 = -12 < 0$

Therefore $x^2 - 2x + 4 = 0$ has 0 real solutions.

Relationship between a quadratic and its reciprocal

Let $y = f(x)$, then the reciprocal of this function is $\frac{1}{y} = \frac{1}{f(x)}$. Given the completed square form of a quadratic function $f(x) = \mu(x - \alpha)^2 + \beta$ we are able to sketch it. From this we are then able to sketch its reciprocal $\frac{1}{f(x)}$. To see how and why first consider this question.

Example (11)

Consider the quadratic function

$f(x) = x^2 + 2$

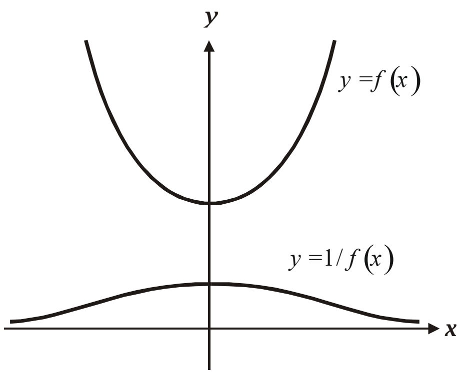
Find the values of $f(x)$ for $x = 0, 1, 2, 3, 4$ and 5. For the same arguments $x = 0, 1, 2, 3, 4$ and 5 find the values of $\frac{1}{f(x)}$. In each case describe what is happening to the sequence of values. What is the relationship between the values of $f(x)$ and the values of $\frac{1}{f(x)}$?

Solution

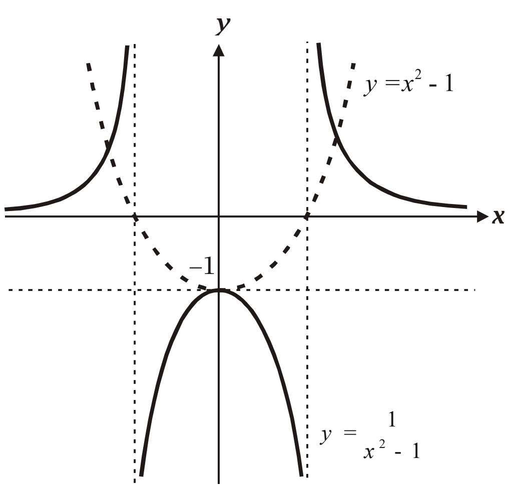
x	0	1	2	3	4	5
$f(x)=x^2+2$	2	3	6	11	18	27
$\frac{1}{f(x)}=\frac{1}{x^2+2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{11}$	$\frac{1}{18}$	$\frac{1}{27}$

For $f(x)=x^2+1$ the sequence of numbers gets larger and larger without limit. For $\frac{1}{f(x)}=\frac{1}{x^2+1}$ the sequence of numbers gets smaller and smaller without limit.

We can sketch the graphs of $f(x)=x^2+1$ and $\frac{1}{f(x)}=\frac{1}{x^2+1}$ to illustrate this relationship.



The graph shows that the minimum of $f(x)=x^2+1$ corresponds to the maximum of $\frac{1}{f(x)}=\frac{1}{x^2+1}$. But note, this is *not* always the case. This works here because the graph of $f(x)=x^2+1$ has no real roots $(\Delta<0)$ and it lies entirely above the *x*-axis. If the function $f(x)$ has real roots then we get a different picture. For example, $f(x)=x^2-1$.



However, let $f(x)$ be a quadratic that has discriminant $\Delta < 0$ and so does not have real roots; let $f(x)$ have a minimum at (α, β); then the *reciprocal* of this function $\frac{1}{y} = \frac{1}{f(x)}$ has a maximum at $\left(\alpha, \frac{1}{\beta}\right)$ - the least value of $y = f(x)$ corresponds to the greatest value of $\frac{1}{y} = \frac{1}{f(x)}$.

Example (9)

(*a*) Express $f(x) = x^2 - 4x + 6$ in the form $(x-a)^2 + b$, where *a* and *b* are constants to be determined. Hence find the least value of $f(x) = x^2 - 4x + 6$ and the corresponding value of *x*.

(*b*) Use the results found in (*a*) to deduce the greatest value of $\frac{1}{f(x)}$.

Solution

(*a*)

$$\begin{aligned} f(x) &= x^2 - 4x + 6 \\ &= \left(x^2 - 4x\right) + 6 \\ &= \left(x^2 - 4x + 2^2 - 2^2\right) + 6 \\ &= \left(x - 4x + 2^2\right) - 4 + 6 \\ &= (x-2)^2 + 2 \end{aligned}$$

The minimum value of $f(x) = x^2 - 4x + 6$ is 2 when $x = 2$.

(*b*) $\frac{1}{f(x)}$ has a maximum at $\left(2, \frac{1}{2}\right)$