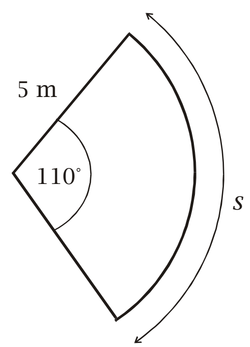
Radian measure

Angles and radians

The division of the circle into 360° is somewhat arbitrary. It stems from Babylonian times, and the number was chosen because many different numbers (factors) divide into it. Mathematicians have an alternative way of describing the angle subtended at the centre of a circle. They measure angles in a unit called *radians*. The usefulness of *radian measure* can be illustrated by means of the problem of finding the *length of an arc of a circle*.

Example (1)

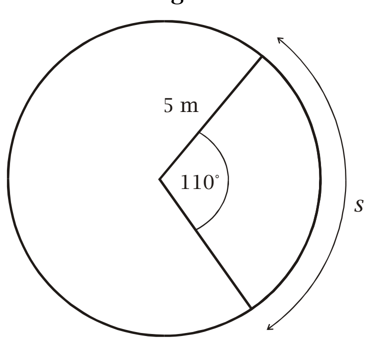
A gentleman owns a rockery in the shape of a segment of a circle.



The rockery subtends an angle of 110° at the centre of the circle, which has a radius of 5 m. He wants to line the outside arc of the rockery with stones and needs to know how long this arc is. Find the length of this arc, shown in the above diagram by *s*.

Solution

We will use trigonometric ratios to solve the problem.



The ratio of the arc length to the whole length is the same as the ratio of the angle subtended by the arc to the whole angle of the circle.

$$\frac{\text{arc length}}{\text{whole circumference}} = \frac{\text{angle subtended}}{\text{whole angle}}$$

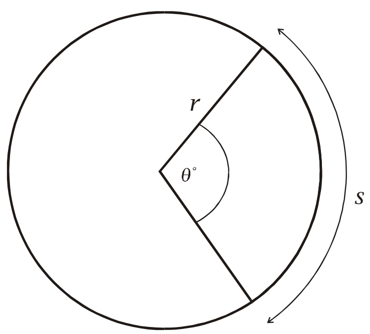
Substituting the relevant values

$$\frac{s}{2\pi r} = \frac{110^\circ}{360^\circ}$$

Given $r = 5\text{m}$

$$s = \frac{110}{360} \times 2\pi \times 5 = 9.60\text{m} \quad (3.\text{s.f.})$$

We can make a general formula out of this. In the following diagram *s* denotes an arc. Suppose we wish to find this length.



To find the length we use ratios.

$$\frac{\text{arc length}}{\text{whole circumference}} = \frac{\text{angle subtended}}{\text{whole angle}}$$

$$\frac{s}{2\pi r} = \frac{\theta}{360^\circ}$$

$$s = \frac{2\pi r\theta}{360}$$

We will now introduce a new measure of angle called the *radian.* Using radians we will obtain a simpler formula for the arc length of a circle. The definition of the radian is based on the following equivalence.

There are 2π radians in a circle.
2π radians is equivalent to 360°.

Example (2)

If the angle subtended by an arc is measured in radians, then the formula for the arc length is simpler. Use the method of ratios above to try to find this formula.

Solution

$$\frac{\text{arc length}}{\text{whole circumference}} = \frac{\text{angle subtended}}{\text{whole angle}}$$

$$\frac{s}{2\pi r} = \frac{\theta}{2\pi}$$

$$s = r\theta$$

This is a simpler formula. We will illustrate its use later, but first we will we familiarise ourselves with the use of radians to measure angles.

Equivalences between angles and radians

The symbol ^{c} indicates that the measure is in radians. The symbol rad is also used. When it is clear from context that the measurement is in radians often no symbol is used at all. Using the symbol ^{c} the definition of a radian is therefore

$$360^\circ \equiv 2\pi^{C}$$

With the symbol rad this is

$$360^\circ \equiv 2\pi \text{ rad}$$

It is important that a student should be able to convert between the two units of measurement with ease. For example 180° is equivalent to π radians.

Example (3)

Find the radian equivalent for the following angles measured in degrees.

(a)	90°	(e)	120°
(b)	45°	(f)	135°
(c)	30°	(g)	270°
(d)	60°	(h)	15°

Solution

To solve this problem, throughout we use the equivalence

$$360^\circ \equiv 2\pi^{C}$$

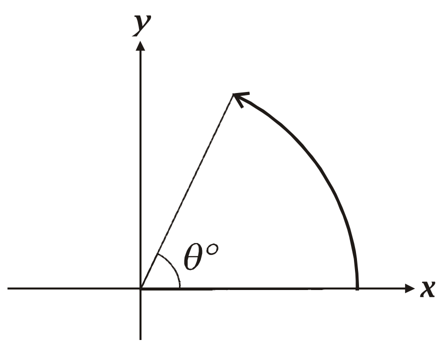
Thus, 90° is $\frac{1}{4}$ of 360°, so 90° is equivalent to $\frac{1}{4}$ of 2π radians, that is, to $\frac{\pi}{2}^{C}$.

Similarly

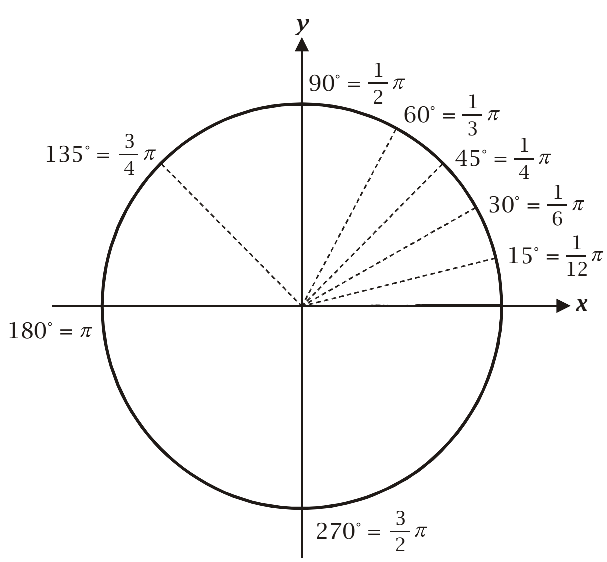
(a)	$90^\circ = \pi/2$	(e)	$120^\circ = 2\pi/3$
(b)	$45^\circ = \pi/4$	(f)	$135^\circ = 3\pi/4$
(c)	$30^\circ = \pi/6$	(g)	$270^\circ = 3\pi/2$
(d)	$60^\circ = \pi/3$	(h)	$15^\circ = \pi/12$

In this solution we have left out the symbol ^{c} because it is clear from context that the angle is measured in radians. It is generally assumed that if θ is an angle and we have an expression such as $\theta = \pi/3$ then θ is measured in radians. This is because radians are defined in terms of π. If, however, the angle is being measured in degrees then we usually put the sign for degrees in.

By convention angles are measured from the positive *x*-axis in an *anticlockwise* direction.



Using this convention we can illustrate the equivalence between radians and angles diagrammatically.



For angles that are not simple fractions of 360 we must convert between degrees and radians using the formulae

$$1^{C} = \frac{360^{\circ}}{2\pi} \qquad 1^{\circ} = \left(\frac{2\pi}{360}\right)^{C}$$

Example (4)

Convert (*a*) 37.3° to radians and (*b*) 0.26 radians to degrees.

Solution

(*a*) $1^{\circ} = \left(\frac{2\pi}{360}\right)^{C}$

$$37.3^{\circ} = \left(\frac{2\pi}{360}\right) \times 37.3 = 0.651^{C} \ (3.\text{s.f.})$$

(*b*) $1^{C} = \frac{360^{\circ}}{2\pi}$

$$0.26^{C} = \frac{360}{2\pi} \times 0.26 = 14.9^{\circ} \ (0.1^{\circ})$$

Arc-length and area of a sector in radians

We have already seen and proven the formula for the arc length *s*.

$s = r\theta$

where the angle θ is measured in radians. The area *A* of a sector is given by

$$A = \frac{1}{2}r^{2}\theta$$

Example (5)

Using the method of ratios prove this formula $A = \frac{1}{2}r^{2}\theta$ for the area *A* of a sector.

$$\frac{\text{area}}{\text{whole area}} = \frac{\text{angle subtended}}{\text{whole angle}}$$

$$\frac{A}{\pi r^{2}} = \frac{\theta}{2\pi}$$

$$\frac{A}{r^{2}} = \frac{\theta}{2}$$

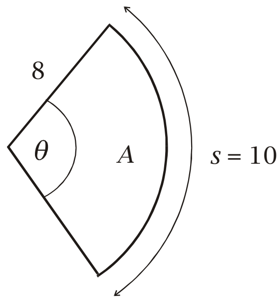
$$A = \frac{1}{2}r^{2}\theta$$

Problem solving involving radians

Example (6)

An arc of a circle has length 10 and radius 8. Find the area subtended by this arc.

Solution



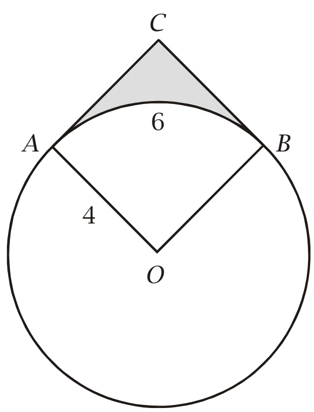
To solve the problem we must first find the angle θ subtended by the arc. To do this we substitute into the formula for the arc length.

$$s = r\theta$$
$$10 = 8\theta$$
$$\theta = \frac{10}{8} = 1.25^{c}$$

Now we know the angle, we can find the area by substituting into the formula for the area.

$$A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 8^2 \times 1.25 = 40 \text{ sq. units}$$

Example (7)



In the diagram the points A and B lie on a circle with centre O and radius 4 cm. The tangents to the circle at A and B intersect at the point C. The length of the arc AB is 6 cm.

(a) Find the angle $A\hat{O}B$

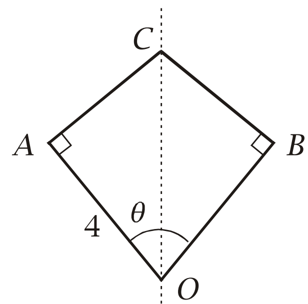
(b) Calculate the area of the shaded region. Give your answer to 3 decimal places.

Solution

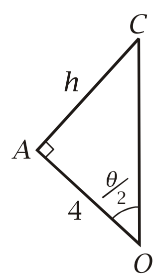
(a) The angle $A\hat{O}B$ may be found by substitution into the formula for arc length.

$$s = r\theta$$
$$6 = 4\theta$$
$$\theta = \frac{6}{4} = 1.5^{\mathrm{c}}$$

(b) The shaded area is the area of the figure $OACB$ less the area of the sector OAB. A tangent to a circle intersects the radius at the point where it touches at right angles. Whilst the diagram makes the figure $OACB$ look like a square, it is not. It is a kite.



To find the area of this kite we need to find its altitude (height), given as h in the following diagram.



$$\tan\frac{\theta}{2} = \frac{h}{4}$$

$$\begin{aligned} h &= 4\tan\left(0.75^{\mathrm{c}}\right) \\ &= 3.7263... \end{aligned}$$

(In order to perform this calculation you must put your calculator in radian mode.)

The area of the kite is

$$\begin{aligned} \text{area}(OACB) &= \text{base} \times \text{height} \\ &= 4 \times 3.7263... \\ &= 14.9055... \end{aligned}$$

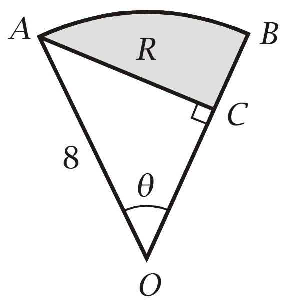
The area of the sector OAB is

$$\text{area}(OAB) = \frac{1}{2}r^2\theta = \frac{1}{2} \times 4^2 \times 1.5 = 12$$

So the area of the shaded region is

$$\begin{aligned} \text{region} &= \text{area}(OACB) - \text{area}(OAB) \\ &= 14.9055... - 12 \\ &= 2.90554... \\ &= 2.91 \text{ (3.s.f.)} \end{aligned}$$

Example (8)

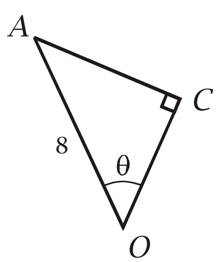


The diagram shows a sector *OAB* of a circle, centre *O* and radius 8 cm. Angle *AOB* is θ radians. The point *C* lies on *OB* and is such that *AC* is perpendicular to *OB*. The region *R* (shaded in the diagram) is bounded by the arc *AB* and by the lines *AC* and *CB*. If the area of *R* is half the area of the triangle *AOC* show that

$$32\vartheta - 48\sin\theta\cos\theta = 0$$

Solution

The area of the region *R* is the area of the sector *OAB* less the area of the triangle *OAC*. The triangle *AOC* is a right-angled triangle with hypotenuse 8, so the lengths *AC* and *OC* can be found by trigonometry.



$$AC = 8\sin\theta$$
$$OC = 8\cos\theta$$

The area of a triangle is

$$\begin{aligned} \triangle &= \frac{1}{2}\text{base} \times \text{perpendicular height} \\ &= \frac{1}{2} 8\sin\vartheta \times 8\cos\vartheta = 32\sin\vartheta\cos\vartheta \end{aligned}$$

The region *R* is

$$R = \text{sector } AOB - \Delta AOC$$
$$= \frac{1}{2}8^2\theta - 32\sin\theta\cos\theta$$
$$= 32\theta - 32\sin\theta\cos\theta \qquad (1)$$

We are told that R is half the area of ΔAOC, so

$$R = \frac{1}{2}\times\Delta AOC = \frac{1}{2}\times 32\sin\theta\cos\theta = 16\sin\theta\cos\theta$$

Substituting for R in equation (1)

$$16\sin\theta\cos\theta = 32\theta - 32\sin\theta\cos\theta$$
$$32\theta - 48\sin\theta\cos\theta = 0$$