Rational Functions and their Decomposition into Partial Fractions

Rational functions

In this context the term *rational function* denotes a function that is expressed as a ratio of any one polynomial to another, each polynomial having rational coefficients. – that is coefficients which can be expressed as fractions. For example

$$f(x) = \frac{x^2 + 3x + 2}{x^3 - 4x^2 + x + 6}$$

is a rational function. As with rational numbers, a rational function is a ratio of a numerator to a denominator. In the case of a rational function, numerator and denominator are functions. In the above example the numerator is $g(x) = x^2 + 3x + 2$ and the denominator is $h(x) = x^3 - 4x^2 + x + 6$.

So the general form of a rational function is

$$f(x) = \frac{g(x)}{h(x)}$$

where $g(x)$ and $h(x)$ are polynomial functions.

Factorisation of the denominator

It is a very important theorem of algebra that every polynomial function can be factorised into factors where each factor is either a linear or at most quadratic factor. That is, every polynomial can be written as:

$$f(x) = g_1(x) \times g_2(x) \times \ldots . \times g_n(x)$$

where each $g_i(x)$ is either a linear factor of the form $(x - \alpha)$ or a quadratic factor of the form $ax^2 + bx + c$. When the factorisation has proceeded as far as it can, any remaining quadratic factors would themselves be incapable of being further decomposed into linear factors with real coefficients. So in such a case they would have negative discriminant $\Delta = b^2 - 4ac < 0$. (These quadratic factors can in fact be further factorised into linear factors with *complex coefficients*; however, in this context we leave them as they are.) It is not immediately obvious that all polynomial functions can be decomposed into linear and quadratic factors (with real coefficients), and the proof of this result, known as the Fundamental Theorem of Algebra, is the subject of higher-level study. In the context of this chapter, the significance is that every denominator of a rational function can be expressed as a multiple of linear and quadratic factors. Here we are also only concerned with denominators that are no more complicated than

$$(ax+b)(cx+d)(ex+f)$$
$$(ax+b)(cx+d)^2$$
$$(ax+b)(x^2+c^2)$$

For instance, the denominator in our example where

$$f(x)=\frac{x^2+3x+2}{x^3-4x^2+x+6}$$

can be decomposed as follows:

$$x^3-4x^2+x+6=(x-3)(x-2)(x+1)$$

The technique of partial fractions

Because denominators can be decomposed it becomes possible to split rational functions into a series of simpler rational functions where the denominators are linear and quadratic functions. This is interesting algebra in its own right, but has practical applications in firstly the sketching of graphs of rational functions, and secondly in finding their integrals. The aim is to rewrite rational functions that cannot directly be integrated into the sum of simpler expressions that can. The technique for doing this is called *decomposition into partial fractions*. Firstly, we rewrite the denominator of the rational function in terms of linear and quadratic factors. The technique will decompose the rational function into a sum of terms with each of these linear and quadratic factors as separate denominators. The result would look something like

$$f(x)=\frac{\cdots}{x-\alpha}+\frac{\cdots}{x-\beta}+\frac{\cdots}{ax^2+bx+c}+\ldots$$

We will demonstrate an algebraic technique for finding the numerators for each part of this expression. In order to find these numerators we work backwards. That is, we assume that the numerator has a certain form and solve some algebra to discover the precise form. Let us begin with a simple example.

Example (1)

Express $\frac{4}{3-2x-x^2}$ as the sum of two partial fractions with linear denominators.

Solution

Firstly, we factorise the denominator

$$\frac{4}{3-2x-x^2}=\frac{4}{(1-x)(3+x)}$$

The next step is the crucial one; we assume that this can now be expressed as two fractions of a specific form, and write

$$\frac{4}{(1-x)(3+x)}\equiv\frac{A}{1-x}+\frac{B}{3+x}$$

where A and B are constants and $A,\ B\neq 0$. We use the equivalence sign, $\equiv$, to express the idea that whatever value x is, the left-hand side of this expression will take the same value as the right-hand side. They are equivalent. Because they are equivalent, we can recombine the right-hand side into a single fraction thus

$$\frac{A}{1-x}+\frac{B}{3+x}\equiv\frac{A(3+x)+B(1-x)}{(1-x)(3+x)}\equiv\frac{3A+Ax+B-Bx}{(1-x)(3+x)}$$

So the original fraction must be equivalent to this

$$\frac{4}{(1-x)(3+x)}\equiv\frac{3A+Ax+B-Bx}{(1-x)(3+x)}$$

Here the denominators are the same, so the numerators must be equivalent.

$$4\equiv 3A+Ax+B-Bx$$

or

$$4\equiv (A-B)x+3A+B$$

The right-hand side of this is true for all values of x, but the left-hand side is a constant function; we could write the left-hand side as

$$4+0x$$

So the coefficients of the x terms on both sides must be equal, and so must the coefficients of the constant terms. We call this process equating coefficients.

$$A-B=0 \qquad (1)$$

$$3A+B=4 \qquad (2)$$

Now these are just two simultaneous equations in two unknowns and we can solve them as follows

$$(1)\Rightarrow A=B$$

$$3A+B=4$$

$$3A+A=4$$

$$4A=4\Rightarrow A=B=1$$

So finally the solution is

$$\frac{4}{(1-x)(3+x)}\equiv\frac{1}{1-x}+\frac{1}{3+x}$$

This is a simple example, but all the other examples follow essentially the same technique; one just needs to recognise the different kind of factors in the denominator and to learn what to do with each. But firstly we will introduce another technique that can simplify the working of the solution.

The Cover-up Rule

This is a labour-saving device. Let us take a second example.

Example (2)

Express $\frac{7x-5}{x^2-x-2}$ as the sum of two partial fractions with linear denominators.

Solution

The solution proceeds as in the first example.

$$\begin{aligned}\frac{7x-5}{x^2-x-2} &\equiv \frac{7x-5}{(x-1)(x-2)} \\ &\equiv \frac{A}{(x+1)} + \frac{B}{(x-2)} \\ &\equiv \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}\end{aligned}$$

Therefore, on equating coefficients

$$7x-5 \equiv A(x-2)+B(x+1)$$

It is at this point that we apply the cover-up rule. This works on the principle that since this expression is true for all values of *x*, we can "cover-up" one of the coefficients by a clever substitution. Thus, on letting $x = 2$ we get

$$7\times 2-5 = A(2-2)+B(2+1)$$
$$9 = 3B$$
$$B = 3$$

The clever substitution has removed the coefficient *A*, so we can skip the tedious process of solving the simultaneous equations. To find *A* we let $x=-1$, then

$$-12 = -3A$$
$$A = 4$$

The technique begins in the same way, but rather than form simultaneous equations in the terms *A*, *B*, … by equating coefficients, we substitute values of *x* that make some of these terms disappear. This is why the method is called the "cover-up" rule, because we effectively substitute values that "cover-up" some of the linear factors. However, the need to solve simultaneous equations cannot always be eliminated. We should also note that at a certain point in the technique we change from the use of an equivalence (≡) to the use of an identity (=) because we start to equate numbers rather than functions.

What to do with each different type of factor

The starting point for the technique is to factorise the denominator and then write the rational function as a sum of fractions involving these factors. We get an expression of the form

$$f(x) = \frac{\cdots}{x-\alpha} + \frac{\cdots}{x-\beta} + \frac{\cdots}{ax^2+bx+c} + \ldots$$

but we need to know how to fill in the spaces represented by the dots here. The answer to this question is merely a set of rules.

Linear factors of the form $(x-\alpha)$

For each linear factor of the form $(x - \alpha)$ in the denominator, write

$$\frac{A}{x-\alpha}$$

in the sum where $A \in \mathbb{R}$ (that is, A is a real number)

Linear factors of the form $(x-\alpha)^2$

For each linear factor of the form $(x-\alpha)^2$ in the denominator write

$$\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2}$$

Two terms are needed because $(x-\alpha)^2 = (x-\alpha)(x-\alpha)$ comprises two factors and each factor generates one coefficient. We cannot write $\frac{\mathrm{A}}{(x-\alpha)}$ twice, and it turns that the second factor in the sum takes the form $\frac{\mathrm{B}}{(x-\alpha)^2}$.

Quadratic factors of the form ax^2+bx+c

For each quadratic factor ax^2+bx+c in the denominator write

$$\frac{Ax+B}{ax^2+bx+c}$$

in the sum.

The technique for finding terms A, B, … is by reforming the sum of the partial fractions into a single fraction over one denominator. Then coefficients in the numerator are equated, and a series of simultaneous equations in the unknown quantities, A, B, … are generated and solved. The cover-up rule may help to simplify the algebra. It is a straightforward technique best learnt through examples.

Example (3)

Express $\dfrac{-x^2 - x - 13}{(x^2+4)(x-1)}$ as the sum of partial fractions.

Solution

$$\frac{-x^2 - x - 13}{(x^2+4)(x-1)} \equiv \frac{Ax+B}{x^2+4} + \frac{C}{x-1}$$

$$\equiv \frac{(Ax+B)(x-1) + C(x^2+4)}{(x^2+4)(x-1)}$$

Therefore

$$-x^2 - x - 13 \equiv (Ax+B)(x-1) + C(x^2+4)$$

$x = 1 \ \Rightarrow -15 = 5C$

$C = -3$

$x = 0 \ \Rightarrow -13 = -B - 12$

$B = 1$

$x = 2 \ \Rightarrow -19 = (2A+1) - 24$

$5 = 2A + 1$

$A = 2$

Hence

$$\frac{-x^2 - x - 13}{(x^2+4)(x-1)} \equiv \frac{2x+1}{x^2+4} - \frac{3}{x-1}$$

Example (4)

Express $\dfrac{x^2 + 5x - 5}{(x-2)^2(x+1)}$ as the sum of partial fractions.

Solution

$$\frac{x^2 + 5x - 5}{(x-2)^2(x+1)} \equiv \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}$$

$$\equiv \frac{A(x-2)(x+1) + B(x+1) + C(x-2)^2}{(x-2)^2(x+1)}$$

$$
\begin{aligned}
&x^2+5x-5 \equiv A(x-2)(x+1)+B(x+1)+C(x-2)^2 \\
&x=2 \Rightarrow \ 9=3B \\
&\qquad\qquad B=3 \\
&x=-1 \Rightarrow -9=9C \\
&\qquad\qquad C=-1 \\
&x=0 \Rightarrow -5=-2A+3-4 \\
&\qquad\qquad -4=-2A \\
&\qquad\qquad A=2
\end{aligned}
$$

$$\frac{x^2+5x-5}{(x-2)^2(x+1)} \equiv \frac{2}{x-2}+\frac{3}{(x-2)^2}-\frac{1}{(x+1)}$$

It may also be necessary to decompose a rational function in the case where the numerator exceeds the denominator. In such cases one simply applies the technique of polynomial division to divide the denominator into the numerator. The remainder is then decomposed into further partial fractions as required.

Example (5)

Decompose $\frac{x^3+2x^2-11x-8}{x^2-2x-3}$ in the sum of partial fractions.

Solution

Here the numerator, $x^3+2x^2-11x-8$, is a polynomial of degree larger than the denominator, so we begin with polynomial division.

$$
\begin{array}{r|l}
 & x+4 \\
x^2-2x-3 & x^3+2x^2-11x-8 \\
 & \underline{x^3-2x^2-3x} \\
 & \qquad 4x^2-8x\ \ -8 \\
 & \qquad \underline{4x^2-8x-12} \\
 & \qquad\qquad\qquad 4
\end{array}
$$

Hence

$$\frac{x^3+2x^2-11x-8}{x^2-2x-3} \equiv x+4+\frac{4}{x^2-2x-3}$$

Then

$$\frac{4}{x^2-2x-3} \equiv \frac{A}{x+1}+\frac{B}{x-3} \equiv \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

Whence by the cover-up rule or otherwise, $A=-1, B=1$ and

$$\frac{x^3+2x^2-11x-8}{x^2-2x-3} \equiv x+4-\frac{1}{x+1}+\frac{1}{x-3}$$