Recurrence relations

Introducing recurrence relations

Example

A investor deposits £1000 in a bank account at 8% interest per annum. How much money will he have after 6 years?

Solution

To solve this problem by the "long" way, we add 8% of the value of the previous year to the value of the previous year, or, equivalently, multiply the value of the previous year by 1.08.

Let u_r = the value of the investment after the rth year.
Then u_0 = the initial investment $\left(\text{called the principal}\right)$ = 1000
u_1 = the value after one year $= 1000 \times 1.08 = 1080$
$u_2 = u_1 \times 1.08 = 1080 \times 1.08 = 1166.4$
$u_3 = u_2 \times 1.08 = 1166.4 \times 1.08 = 1259.712$
$u_4 = u_3 \times 1.08 = 1259.712 \times 1.08 = 1360.48896$
$u_5 = u_4 \times 1.08 = 1360.48896 \times 1.08 = 1469.328077$
$u_6 = u_5 \times 1.08 = 1469.328077 \times 1.08 = 1586.874323$
$u_6 = 1586.87 \left(\text{to the nearest penny}\right)$

This is an example of a recurrence relation – that is, a relation where the value at one stage of the computation is calculated in terms of one more values at earlier stages.

The recurrence relation here is

$$u_{r+1} = u_r \times 1.08$$

That is, the value of the investment at the end of the $\left(r+1\right)th$ year is given by the value at the end of the rth year × 1.

There is a "shorter" method of finding the solution to this problem, expressed by the relation

$u_r = u_0 \times (1.08)^r$

That is,

$u_6 = 1000 \times (1.08)^6 = 1586.874323 = 1586.87$ (to the nearest penny)

This topic is concerned with finding these "short-cut" solutions to problems that are expressed in terms of recurrence relations. We may express this solution to this problem in general terms as

Given the recurrence relations
$u_r = u_0 \times k$
Then
$u_r = u_0 \times k^r$

Example

Write down the recurrence relation corresponding to the following situation: an investor deposits £1000 in a bank account at 8% interest per annum. At the end of each year he adds a further £300 to his deposit.

Solution

The expression for the recurrence relation in this case is

$u_{r+1} = 1.08u_r + 300$

Classification of recurrence relations

The order of a recurrence relation

The order of a recurrence relation is the difference the highest and lowest subscripts of the terms used in the relationship. For example, in the relationship

$u_{r+1} = 4u_r - 5$

the difference between the highest subscript, u_{r+1}, and the lowest subscript, u_r is
$(r+1) - r = 1$
so the order of the recurrence relation is 1.

In the relationship

$$u_{r+1} = u_r + 2u_{r-1} + 2$$

the difference between the highest and lowest subscript is

$$(r+1)-(r-1)=2$$

therefore the order of the relation is 2.

<u>Linear and non-linear recurrence relations</u>

A recurrence relation is linear if the coefficients in the relation are not functions of the previous values of the relation. In other words, in the relation

$$u_{r+1} = a_r u_r + b_r u_{r-1} + c_r u_{r-2} + ... + k_r$$

the coefficients, a_r, b_r, c_r may be functions of r by they may not be functions of any of the terms u_r, u_{r-1}, u_{r-2}, ...

A recurrence relation that does not satisfy this condition is said to be non-linear.

<u>Linear, constant-coefficient recurrence relation</u>

This is a linear recurrence relation where the coefficients are constants – that is, they are not functions of *r*.

<u>Homogenous and non-homogeneous</u>

If the term k_r in the linear recurrence relation

$$u_{r+1} = a_r u_r + b_r u_{r-1} + c_r u_{r-2} + ... + k_r$$

is 0, then the relation is said to be homogenous. When k_r is not 0, then the relation is non-homogeneous.

Thus, for example, the recurrence relation $u_{r+2} = u_{r-1} \times u_{r-2}$ is not linear; the recurrence relation $u_r = r^2 u_{r-1}$ is linear but not constant coefficient and $u_{r+1} = u_r + 2u_{r-1} + 2$ is second order, linear and constant coefficient, but it is non-homogeneous. The relation $u_{r+1} = u_r + 2u_{r-1}$ is second order, linear, constant-coefficient and homogenous.

Solution of first-order, constant-coefficient linear recurrence relations

First-order, constant-coefficient, linear and homogeneous recurrence relations

These are of the form

$$u_{r+1} = au_r$$

This is the simplest form of recurrence relation, and its solution is

$$u_n = a^n u_0$$

where u_0 is the initial value.

Example

A recurrence relation is given by

$$u_{r+1} = 3u_r$$

If the initial value is 6, find u_7

Solution

$$u_n = a^n u_0$$

Here $a = 3,\ u_0 = 6,$ hence,

$$u_n = 6 \times 3^7 = 13122$$

The proof of the formula depends on knowledge of mathematical induction. As this knowledge is not assumed here, this is left to another chapter.

Even more generally, the first-order, linear, constant-coefficient, homogeneous recurrence relation

$$u_{r+1} = au_r$$

has general solution

$$u_n = B \times a^n$$

where B is a constant

The value of the constant for a particular solution is found by substituting a particular value for which u_n is known.

First-order, constant-coefficient, linear and inhomogeneous recurrence relations

These have general form

$$u_{r+1} = au_r + k$$

and has general solution

$$u_n = Ba^n - \frac{k}{a-1} \qquad \text{if } a \neq 1$$

and

$$u_n = A + nk \qquad \text{if } a = 1$$

Example

Find the general solution to the recurrence relation

$$u_{r+1} = 3u_r - 2$$

and the particular solution if $u_0 = 2$.

Solution

The general solution is

$$u_n = Ba^n - \frac{k}{a-1}$$

where $a = 3$ and $k = -2$

hence,

$$u_n = B3^n - \frac{(-2)}{3-1}$$

Therefore,

$$u_n = B3^n + 1$$

To find the particular solution we substitute, $u_0 = 2,\ n = 0$ to obtain

$2 = B3^0 + 1$

$B = 1$

Hence,

$u_n = 3^n + 1$

is the particular solution.

The proof of the formula for first-order, constant-coefficient, linear and inhomogeneous recurrence relations requires knowledge of geometric progressions. As that is not assumed here, it is left to another chapter.

Linear, second-order recurrence relations

A linear, second-order, constant-coefficient homogeneous recurrence relation has the form

$u_{r+1} = au_r + bu_{r-1}$

where a and b are constants.

To solve this, form the auxiliary equation

$m^2 = am + b$

Rearrange it to give

$m^2 - am - b = 0$

Solve this quadratic equation to give roots λ and μ.

Then, the solution will depend on the number and type of roots. If

(1) There are different roots, so that $\lambda \neq \mu,$ then

$u_n = A\lambda^n + B\mu^n$

where A and B are constants.

(2) There is one single roots, so that $\lambda = \mu$

Then the solution is

$u_n = (A + Bn)\lambda^n$

(3) There are no real roots, but complex roots, then the auxiliary equation as two complex, conjugate roots which can be written in polar form

$\lambda = r(\cos\theta + i\sin\theta)$ and $\mu = r(\cos\theta - i\sin\theta)$

Then the general solution to the recurrence relation is

$u_n = r^n (A\cos n\theta + B\sin n\theta)$

where A and B are constants.

Example

Find the particular solution to the recurrence relation

$u_{r+1} = 6u_r - 8u_{r-1}$

when $u_0 = 4$ and $u_1 = 6$

The auxiliary equation is

$m^2 = 6m - 8$

or

$m^2 - 6m + 8 = 0$

with solution

$(m-4)(m-2) = 0$

giving

$\lambda = 4,\ \mu = 2$

Hence, since there are different roots, so that $\lambda \neq \mu$, then $u_n = A\lambda^n + B\mu^n$ and

$u_n = A4^n + B2^n$

To find the particular solution, we substitute $u_0 = 4,\ n = 0$ and $u_1 = 6,\ n = 1$ in turn

$4 = A + B \quad (1)$

$6 = 4A + 2B \quad (2)$

Solving these two equations simultaneously

$(1)\times 2$ gives $\quad 8 = 2A + 2B \quad (3)$

$(2)-(3) \qquad -2 = 2A$

$A = -1$

$B = 5$

Hence, the particular solution is

$u_n = -4^n + 5\times 2^n$

Example

Solve the recurrence relation

$u_{r+1} = 2u_r - 2u_{r-1}$

with initial conditions,

$u_0 = 1,\ u_1 = \sqrt{2}$

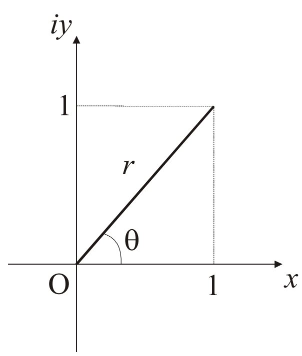
The auxiliary equation is

$m^2 = 2m - 2$

$m^2 - 2m + 2$

with solution

$\lambda, \mu = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$



Hence, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}(1) = \pi/4$

Therefore,

$\lambda, u = \sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right)$

And so,

$u_n = \left(\sqrt{2}\right)^n \left(A \cos n\left(\pi/4\right) + B \sin n\left(\pi/4\right)\right)$

$u_n = \left(\sqrt{2}\right)^n \left(A \cos n\left(\pi/4\right) + B \sin n\left(\pi/4\right)\right)$

To find the particular solution, we substitute $n = 0,\ u_0 = 1$ and $n = 1,\ u_1 = \sqrt{2},$ then

$$1 = \left(\sqrt{2}\right)^0 \left(A\cos 0\left(\pi/4\right) + B\sin 0\left(\pi/4\right)\right)$$

$$A = 1$$

and

$$2 = \left(\sqrt{2}\right)^1 \left(A\cos \pi/4 + B\sin \pi/4\right)$$

$$2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + B\frac{1}{\sqrt{2}}\right)$$

$$2 = 1 + B$$

$$B = 1$$

Thus, the particular solution is

$$u_n = \left(\sqrt{2}\right)^n \left(\cos n\left(\pi/4\right) + \sin n\left(\pi/4\right)\right)$$