Reduction formulae

Pre-requisites

You should be familiar with the technique of integration by parts where the integration must be performed twice as in the following example.

Example (1)

Use integration by parts to find $\int e^{2x} \cos x \; dx$

Solution

Let

$f(x) = e^{2x}$ $\qquad\qquad g'(x) = \cos x$

Then

$f'(x) = 2e^{2x}$ $\qquad\qquad g(x) = \sin x$

The integration by parts formula is

$\int fg' = fg - \int f'g$

Substitution into it gives

$\int e^{2x} \cos x \; dx = \text{-}e^{2x} \sin x - \int 2e^{2x} \sin x \; dx$

We can take the 2 on the right-hand-side outside the integral side

(1) $\qquad \int e^{2x} \cos x \; dx = -e^{2x} \sin x - 2\int e^{2x} \sin x \; dx$

However, we now need to find $\int e^{2x} \sin x \, dx$. This requires a second application of the integration by parts formula. In this case

$f(x) = e^{2x}$ $\qquad\qquad g'(x) = \sin x$

$f'(x) = 2e^{2x}$ $\qquad\qquad g(x) = -\cos x$

Then

$\int e^{2x} \sin x \; dx = -e^{2x} \cos x - \int 2e^{2x} \cos x \; dx$

Substituting for $\int e^{2x} \sin x \; dx$ at (1) gives

$\int e^{2x} \cos x \; dx = e^{2x} \sin x - 2\left\{\text{-}e^{2x} \cos x + 2\int e^{2x} \cos x \; dx\right\}$

$\therefore \int e^{2x} \cos x \; dx = e^{2x} \sin x + 2e^{2x} \cos x - 4\int e^{2x} \cos x dx$

Collecting the terms in $\int e^{2x} \cos x \; dx$ gives

$5\int e^{2x} \cos x \; dx = e^{2x} \sin x + 2e^{2x} \cos x$

$\therefore \int e^{2x} \cos x \; dx = \frac{1}{5} e^{2x} (\sin x + 2\cos x)$

Reduction formulae

We find $\int \cos^2 x \, dx$ by means of the trigonometric identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

Example (2)

Find $\int \cos^2 x \, dx$

Solution

$$\begin{aligned}\int \cos^2 x \, dx &= \int \frac{1}{2} + \frac{1}{2}\cos 2x.dx \\ &= \frac{x}{2} + \frac{\sin 2x}{4} + c \\ &= \frac{x}{2} + \frac{\sin x \cos x}{2} + c\end{aligned}$$

For higher powers of $\cos^n x$, i.e. for $n > 4$, the use of trigonometric identities becomes tedious. Fortunately, a further technique exists – the use of a reduction formula. When a reduction formula is used we rewrite the integral in, say, $\cos^n x$ in terms of some power less than n. We then repeat the process. The method is best illustrated by example.

Example (3)

Find

$I = \int \cos^4 x \, dx$

Solution

We integrate $\cos^4 x$ by parts. The parts formula is:-

$$\int fg' = fg - \int f'g$$

Letting $f = \cos^3 x$ then

$$\begin{aligned}f' &= -3\cos^2 x \sin x \\ g' &= \cos x \\ g &= \sin x\end{aligned}$$

On substitution

$$\begin{aligned}I &= \cos^3 x \sin x - \int -3\cos^2 x \sin x \sin x.dx \\ &= \cos^3 x \sin x + 3\int \cos^2 x \sin^2 x.dx \\ &= \cos^3 x \sin x + 3\int \cos^2 x \left(1 - \cos^2 x\right).dx \\ &= \cos^3 x \sin x + 3\int \cos^2 x.dx - 3\int \cos^4 x.dx\end{aligned}$$

But $\int \cos^4 x.dx = I$ so collecting terms we get

$$4I = \cos^3 x \sin x + 3\int \cos^2 x \, dx$$

Then dividing by 4 and substituting from the previous question for $\int \cos^2 x \; dx$

$$\int \cos^4 x \; dx = \int \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{x}{2} + \frac{\cos x \sin x}{2}\right)$$
$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + c$$

What this shows is the iteration of a process at each stage reducing the problem to one with a lower exponent. Here the integration by parts formula was applied so as to write the $\int \cos^4 x \; dx$ integral in terms of an integral involving $\int \cos^2 x \; dx$ - that is, with an exponent 2 less than the original. The parts formula was iterated and the solution obtained. We continue this investigation by examining $\int \cos^6 x \; dx$.

Example (4)

Find $\int \cos^6 x \; dx$

Solution

As before, let $I = \int \cos^6 x \; dx,$ and we will being by integrating by parts

$$\int fg' = fg - \int f'g$$

$$f = \cos^5 x$$
$$f' = -5\cos^5 x$$
$$g' = \cos x$$
$$g = \sin x$$

Therefore

$$I = \cos^5 x \sin x - \int -5\cos^4 x \sin x \sin x.dx$$
$$= \cos^5 x \sin x + 5\int \cos^4 x \sin^2 x.dx$$
$$= \cos^5 x \sin x + 5\int \cos^4 x\left(1 - \cos^2 x\right).dx$$
$$= \cos^5 x \sin x + 5\int \cos^4 x.dx - 5\int \cos^6 x.dx$$

Substituing $I = \int \cos^6 x \; dx$ and collecting:-

$$6I = \cos^5 x \sin x + 5\int \cos^4 x \; dx$$

Dividing by 6 and substituting for $\int \cos^4 x \; dx$

$$\int \cos^6 x \; dx = \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\left(\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3\times 1}{4\times 2}x\right) + c$$
$$= \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{16}\cos x \sin x + \frac{5\times 3\times 1}{6\times 4\times 2}x + c$$

Let us summarise what we have discovered so far regarding $\int \cos^n x \; dx$

For $n=2$ $\int \cos^2 x\, dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + c$

For $n=4$ $\int \cos^4 x\ dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\sin x \cos x + \frac{3\times 1}{4\times 2}x + c$

For $n=6$ $\int \cos^6 x\ dx = \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5}{16}\cos x \sin x + \frac{5\times 3\times 1}{6\times 4\times 2}x + c$

Let us now examine the definite integrals of the form $\int_0^{\pi/2} \cos^n x\ dx$

For $n=2$ $I_2 = \int_0^{\pi/2} \cos^2 x\ dx = \left[\frac{1}{2}\cos x \cdot \sin x + \frac{1}{2}x\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\times\frac{\pi}{2}$

For $n=4$ $I_4 = \int_0^{\pi/2} \cos^4 x\ dx = \frac{3\times 1}{4\times 2}\times\frac{\pi}{2} = \frac{3}{4}I_2$

For $n=6$ $I_6 = \int_0^{\pi/2} \cos^6 x\ dx = \frac{5\times 3\times 1}{6\times 4\times 2}\times\frac{\pi}{2} = \frac{5}{6}I_4$

From this we can form the conjecture:

$$I_n = \frac{n-1}{n}\times I_{n-2} \quad n \ge 2$$

This is an example of a reduction formula because it evaluates an integral of a certain type in an expression to the power n in terms of an integral of the same type but in a reduced power. Repeated applications of a reduction formula will eventually reduce an integral to be found to a known integral. Thus, in this example

$$I_2 = \frac{1}{2}\times\frac{\pi}{2}$$

$$I_6 = \frac{5}{6}I_4 = \frac{5}{6}\times\frac{3}{4}I_2 = \frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{5\pi}{32}$$

We will now prove the validity of the conjecture.

Let $I_n = \int_0^{\pi/2} \cos^n x\ dx$

We will now integrate this by parts.

$$\int f\cdot g' = f\cdot g - \int f'\cdot g$$

$$f = \cos^{n-1} x$$

$$f' = -(n-1)\cos^{n-2} x \sin x$$

$$g' = \cos x$$

$$g = \sin x$$

$$I_n = \left[\sin x \cos^{-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} -(n-1)\cos^{n-2} x \cdot \sin^2 x \cdot dx$$

$$= 0 + (n-1)\int_0^{\pi/2} \cos^{n-2} x\left(1-\cos^2 x\right)\cdot dx$$

$$= (n-1)\int_0^{\pi/2} \cos^{n-2} x \cdot dx - (n-1)\int_0^{\pi/2} \cos^n x \cdot dx$$

$$= (n-1)\int_0^{\pi/2} \cos^{n-2} x \cdot dx - (n-1)I_n$$

$$\therefore nI_n = (n-1)\int_0^{\pi/2} \cos^{n-2} x \cdot dx$$
$$\therefore I_n = \frac{n-1}{n} \cdot I_{n-2} \qquad n \geq 2$$

Questions on reduction formula ask you to prove such relationships as

$$I_n = \frac{n-1}{n} \cdot I_{n-2} \quad n \geq 2$$

It is a process that will generally involve integration by parts. The question will then ask you to apply the result to find a definite integral, as in

$$I_6 = \frac{5}{6} I_4 = \frac{5}{6} \cdot \frac{3}{4} \cdot I_2 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{32}$$

Example (5)

(i) If $I_n = \int \operatorname{cosec}^n(x)\,dx$, prove that

$$I_n = -\frac{1}{n-1}\cot(x)\operatorname{cosec}^{n-2}(x) + \frac{n-2}{n-1} I_{n-2}, \quad (n \geq 2)$$

and use this reduction formula to write down an expression relating I_n to I_{n-4} $(n \geq 4)$.

(ii) Use the results obtained in (i) to find $\int_{\pi/6}^{\pi/2} \operatorname{cosec}^4(x)\,dx$.

(iii) Prove that $\int_{\pi/3}^{\pi/2} \operatorname{cosec}^3(x)\,dx = \frac{1}{3} + \frac{\ln(3)}{4}$.

Solution

(i) $I_n = \int \operatorname{cosec}^n(x)\,dx$

$$I_{n-2} = \int \frac{1}{\sin^{n-2}(x)}\,dx = \int \frac{1-\cos^2(x)}{\sin^n(x)}\,dx$$
$$= I_n - \int \cos(x) \frac{d(\sin(x))}{\sin^n(x)}$$
$$= I_n - \left[\frac{\cos(x)}{(-n+1)\sin^{n-1}(x)} - \frac{1}{-n+1} \int \frac{1}{\sin^{n-1}(x)} (-\sin(x))\,dx \right]$$

so $I_{n-2} = I_n + \frac{\cos(x)}{(-n+1)\sin^{n-1}(x)} + \frac{1}{n-1} I_{n-2}$

Hence

$$I_n = I_{n-2} = \frac{\cos(x)}{(n-1)\sin^{n-1}(x)} + \frac{n-2}{n-1} I_{n-2}$$

So $\int \operatorname{cosec}^n(x)\,dx = -\frac{1}{n-1}\cot(x)\operatorname{cosec}^{n-2}(x) + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2}(x)\,dx, \quad (n \geq 2)$

$$I_n = -\frac{1}{n-1}\cot(x)\operatorname{cosec}^{n-2}(x) + \frac{n-2}{n-1}\left[-\frac{1}{n-3}\cot(x)\operatorname{cosec}^{n-4}(x) + \frac{n-4}{n-3} I_{n-4} \right]$$

(ii) $$\int_{\pi/6}^{\pi/2} \operatorname{cosec}^4(x)\,dx = \left[-\frac{1}{3}\cot(x)\operatorname{cosec}^2(x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \frac{2}{3}\int_{\pi/6}^{\pi/2} \operatorname{cosec}^2(x)\,dx$$

$$= \frac{1}{3}\sqrt{3}\times 4 + \frac{2}{3}\left[-\cot(x)\operatorname{cosec}^2(x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{4}{3}\sqrt{3} + \frac{2}{3}\left(\sqrt{3}\right)$$

$$= \frac{6\sqrt{3}}{3}$$

$$= 2\sqrt{3}$$

(iii) $$\int_{\pi/3}^{\pi/2} \operatorname{cosec}^3(x)\,dx = \left[\left(-\frac{1}{2}\right)\cot(x)\frac{1}{\sin(x)}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \frac{1}{2}\int_{\pi/3}^{\pi/2} \frac{1}{\sin(x)}\,dx$$

$$= -\frac{1}{2}\left(0 - \frac{2}{\sqrt{3}\cdot\sqrt{3}}\right) + \frac{1}{2}\left[\ln\left|\tan\left(\frac{x}{2}\right)\right|\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{1}{3} + \frac{1}{2}\left(\ln(1) + \ln\left(\sqrt{3}\right)\right) = \frac{1}{3} + \frac{\ln(3)}{4}$$

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