Remainder and Factor Theorems

Prerequisites - polynomial division

You should be familiar with the process of polynomial division.

Example (1)

Use polynomial division to find $\left(x^3+2x^2-x+3\right)\div(x-1)$

Solution

$$\begin{array}{r|l} & x^2+3x+2 \\ \hline x-1 & x^3+2x^2-x+3 \\ & \underline{x^3-x^2} \\ & \quad 3x^2-x \\ & \quad \underline{3x^2-3x} \\ & \qquad 2x+3 \\ & \qquad \underline{2x-2} \\ & \qquad\quad 5 \end{array}$$

$$\therefore x^3+2x^2-x+3=(x-1)\left(x^2+3x+2\right)+5$$

Remainder theorem

Suppose we wish to find the remainder when one polynomial $f(x)$ is divided by another $g(x)$. The remainder theorem provides a short cut to this remainder without the use of polynomial division. Notice that the general form of a polynomial division looks like

$$\begin{array}{r|l} & \quad g(x) \\ \hline (x-\alpha) & \quad f(x) \\ & \quad \vdots \\ & \underline{\qquad\qquad} \\ & \qquad R \end{array}$$

where $f(x)$ is the dividend, $(x-\alpha)$ is the divisor, $g(x)$ is the quotient, and R is the remainder. We can also write this as

$$f(x)=(x-\alpha)g(x)+R$$

Example (2)

Find the value of $f(x) = (x-\alpha)g(x) + R$ when $x = \alpha$.

Solution

This question is asking you to substitute $x = \alpha$ in both sides of $f(x) = (x-\alpha)g(x) + R$

$$f(x) = (x-\alpha)g(x) + R$$
$$f(\alpha) = (\alpha-\alpha)g(\alpha) + R$$
$$f(\alpha) = R$$

This is the *Remainder Theorem*, which states that by substituting α for x in $f(x)$ we obtain the remainder when dividing $f(x)$ by $(x - \alpha)$.

Example (3)

Find the remainder when $(x^2 + 3x - 2)$ is divided by $(x - 5)$.

Solution

Let $f(x) = x^2 + 3x - 2$

Then, $f(5) = 25 + 15 - 2 = 38$

So the remainder is 38.

Problems involving the remainder theorem

Problems are set in which the student is required to find an unknown quantity.

Example (4)

When $x^3 - 3x^2 + bx - 1$ is divided by $x - 2$ the remainder is 5. Find *b*.

Solution

Let $f(x) = x^3 - 3x^2 + bx - 1$

Then, $f(2) = 8 - 12 + 2b - 1 = 5$

$\therefore \quad 2b = 10$

$b = 5$

The factor theorem

A *corollary* is a further result derived from a theorem. The Remainder theorem has a useful corollary. This is the *Factor theorem*. You should know that a factor of a polynomial $f(x)$ is another polynomial $x-\alpha$ that divides completely into $f(x)$ without remainder. If $x-\alpha$ is a factor of $f(x)$ then α is a root of $f(x)$. The factor theorem enables us to quickly search for roots to polynomials.

The Remainder theorem states that $f(\alpha)=R$ when $f(x)$ is divided by $(x-\alpha)$. If $x-\alpha$ is a factor of $f(x)$ then the remainder $R=0$. From the Remainder theorem this implies $f(\alpha)=0$. Putting this the other way around, if $f(\alpha)=0$ then $x-\alpha$ is a factor of $f(x)$ and α is one of its roots.

Remainder Theorem

$f(\alpha)=0$ implies that $(x-\alpha)$ is a factor of $f(x)$

and

$(x-\alpha)$ is a factor of $f(x)$ implies that $f(\alpha)=0$

Example (5)

Given that x^3-3x^2+ax+b is divisible by both $x-2$ and $x-3$, find a and b.

Solution

Let $f(x)=x^3-3x^2+ax+b$

Since $(x-2)$ is a factor of $f(x)$, then $f(2)=0$

Then, $f(2)=8-12+2a+b=0$

$\therefore \quad 2a+b=4$

Since $(x-3)$ is a factor of $f(x)$, then $f(3)=0$

$f(3)=27-27+3a+b=0$

$\therefore \quad 3a+b=0$

Solving simultaneously

$a=-4$

$b=12$

Factorising polynomials

We once again remind you that a linear factor of a polynomial $f(x)$ is another polynomial $x-\alpha$ that divides completely into $f(x)$ without remainder. To say that a polynomial $f(x)$ can be completely factorised is to say that it can be written as the product of linear factors. Not all polynomials can be completely factorised, but if factors do exist then the Factor theorem can be used to find them.

So one use of the Factor theorem is to assist in the process of factorising polynomials. For example, a cubic equation has the form

$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$

You may be asked to factorise a cubic equation completely. In such as case

1. Search for one root using the Factor theorem.
2. Look for a quadratic factor by using polynomial division. If the discriminant,

   $\Delta = b^2 - 4ac$

   is larger than zero, then factorise the quadratic factor. If it is not greater than zero, then the quadratic factor does not factorise.

The following example requires you to factorise completely a quartic polynomial, that is a polynomial beginning with an x^4 term. However, this problem quickly resolves into a sub-problem requiring the factorisation of a cubic polynomial. To solve the problem for the cubic you must begin by using the factor theorem to search for a root. Once you have one root you can divide the cubic by the corresponding factor using polynomial division to obtain a quotient that is a quadratic. In this particular case the quadratic can be factorised. So finally, the original function is fully factorised into four linear factors.

Example (6)

Factorise completely $6x^4 - 23x^3 + 11x^2 + 12x$

Solution

$6x^4 - 23x^3 + 11x^2 + 12x = x\left(6x^3 - 23x^2 + 11x + 12\right)$

So we need to search for a factor of the cubic polynomial

$g(x) = 6x^3 - 23x^2 + 11x + 12$

To find this factor we must search for it systematically using the factor theorem. This means to substitute progressively different values for x in $g(x)$ until the result is zero.

$g(0)=12$

$g(1)=6-23+11+12=6$

$g(-1)=-6-23-11+12=-28$

$g(2)=48-92+22+12=-10$

$g(-2)=-48-92-22+12=-150$

$g(3)=162-207+33+12=0$

So we have obtained a value of $g(x)$ that is zero, so by the factor theorem $(x-3)$ is a factor of $g(x)$. We need to find the other factor of $g(x)$. To find this proceed to divide $g(x)$ by $(x-3)$.

$$\begin{array}{r} 6x^2-5x-4 \\ x-3\overline{)6x^3-23x^2+11x+12} \\ \underline{6x^3-18x^2} \qquad\qquad \\ -5x^2+11x \quad \\ \underline{-5x^2+15x} \quad \\ -4x+12 \\ \underline{-4x+12} \end{array}$$

This tells us that $g(x)=(x-3)(6x^2-5x-4)$. Here $6x^2-5x-4$ is a quadratic polynomial. Its discriminant is

$\Delta=b^2-4ac=25+96=121>0$

which tells us that it has two real roots. It may be factorised by inspection or by means of the quadratic formula to give.

$6x^2-5x+4=(3x-4)(2x+1)$

Hence

$$\begin{aligned} 6x^4-23x^3+11x^2+12x &= x\left(6x^3-23x^2+11x+12\right) \\ &= x(x-3)\left(6x^2-5x+4\right) \\ &= x(x-3)(3x-4)(2x+1) \end{aligned}$$

The originally polynomial has been fully factorised into linear factors.