Richardson's Extrapolation

If a numerical technique is used to approximate an integral, a better approximation can be obtained by using more and more strips. However, there is a snag which occurs when using this approach to improving accuracy when using machines. Any machine can only carry computations to a certain level of accuracy. They "chop off" numbers at a certain point and this leads to a rounding error.

For example, when the fraction, $\frac{1}{3} = 0.33333......$

is rounded to 4 significant figures to 0.3333

this leads to a rounding error of

$$\frac{1}{3} - 0.3333 = 0.0000333....$$

Such rounding errors may be insignificant in themselves, but repeated over thousands of calculations can introduce considerable error. It can be shown that when the trapezium and Simpson's rules are used the number of rounding error due to the calculation for each value of $f(x)$ does increase as the number of strips increases. In other words there is an an upper limit to how well an integral can be approximated by these techniques, since reducing the error due to the width of the interval increases the error due to rounding off.

Recall that the principal error in the trapezium rule method is proportional to h^2 and in Simpson's method it is proportional to h^4, where h is the width of the interval. The Richardson extrapolation improves the principal error in the trapezium method to h^4 and in the Simpson's method to h^6 without increasing the rounding error significantly. In both cases the method is based upon taking two approximations to the interval, one based on the n strips and the other based on $2n$ strips.

For both methods, let

S_n represent the approximation based on the n strips with width h

S_{2n} represent the approximation based on the $2n$ strips with width $\frac{h}{2}$

Then for the Trapezium method the interval is approximated by

$S = \frac{4}{3} S_{2n} - \frac{1}{3} S_n$

The principal error is proportional to h^4 (or "of the order of h^4").

For Simpson's Method

$S = \frac{16}{15} S_{2n} - \frac{1}{15} S_n$

The principal error is the order of h^6 (is proportional to h^6).

Example

Find approximations to $I = \int_{\pi/6}^{\pi/3} \frac{1}{\sin x} \delta x$

By means of the trapezium rule with 3 and 6 intervals. Use Richardson's extrapolation to improve on both approximations. Use 6 decimal places in your calculations. Given

$\int \operatorname{cosec} x \delta x = \ln\left(\tan \frac{x}{2}\right)$

find the exact value for I and the error in all three approximations.

x	$y(x)\frac{1}{\sin x}$
$\pi/6$	2
$7\pi/36$	1.743447
$8\pi/36 = 2\pi/9$	1.555724
$9\pi/36 = \pi/4$	1.414214
$10\pi/36 = 5\pi/18$	1.305407
$11\pi/36$	1.220775
$12\pi/36 = \pi/3$	1.154701

The trapezium method is

$$I \approx \tfrac{h}{2}\left(y_0 + y_n + +2\left(y_1 + ... + y_{n-1}\right)\right)$$

For $n = 3$ the approximation is

$$\begin{aligned} S_3 &= \tfrac{\pi}{36}\left[y\left(\tfrac{\pi}{6}\right) + y\left(\tfrac{\pi}{3}\right) + 2\left(y\left(\tfrac{2\pi}{9}\right) + y\left(\tfrac{5\pi}{18}\right)\right)\right] \\ &= \tfrac{\pi}{36}\left(2 + 1.154701 + 2\left(1.555724 + 1.305407\right)\right) \\ &= 0.774661 \end{aligned}$$

For $n = 6$

$$\begin{aligned} S_6 &= \tfrac{\pi}{72}\left[y\left(\tfrac{\pi}{6}\right) + y\left(\tfrac{\pi}{3}\right) + 2\left(y\left(\tfrac{7\pi}{36}\right) + y\left(\tfrac{2\pi}{9}\right) + y\left(\tfrac{\pi}{4}\right) + y\left(\tfrac{5\pi}{18}\right) + y\left(\tfrac{11\pi}{36}\right)\right)\right] \\ &= \tfrac{\pi}{72}\left[2 + 1.154701 + 2\left(1.743447 + 1.555724 + 1.414214 + 1.305407 + 1.220775\right)\right] \\ &= 0.769421 \end{aligned}$$

Richardson's extrapolation is

$$\begin{aligned} S &= \tfrac{4}{3}S_{2n} - \tfrac{1}{3}S_n = \tfrac{4}{3}S_6 - \tfrac{1}{3}S_3 \\ &= \left(\tfrac{4}{3}\times 0.769421\right) - \left(\tfrac{1}{3}\times 0.774661\right) \\ &= 0.767674 \end{aligned}$$

We have $\int \cos\text{ec}\, x \delta x = \ln\left(\tan\tfrac{x}{2}\right)$

Therefore

$$\begin{aligned} I &= \int_{\pi/6}^{\pi/3} \frac{1}{\sin x} \bullet \delta x = \int_{\pi/6}^{\pi/3} \cos\text{ec} x \bullet \delta x \\ &= \left[\ln\left(\tan\tfrac{\pi}{2}\right)\right]_{\pi/6}^{\pi/3} \\ &= \ln\left(\tan\tfrac{\pi}{6}\right) - \ln\left(\tan\tfrac{\pi}{12}\right) \\ &= 0.767652 \end{aligned}$$

$$\begin{aligned}\left|I - S_3\right| &= \left|0.767652 - 0.769421\right| \\ &= 0.007009 \\ \left|I - S_6\right| &= \left|0.767652 - 0.769421\right| \\ &= 0.001769 \\ \left|I - S\right| &= \left|0.767652 - 0.767674\right| \\ &= 0.000022\end{aligned}$$

This illustrates that even with the Trapezium method and only 6 intervals Richardson's extrapolation produces a good approximation.