Roots and Coefficients of Quadratic Polynomials

A quadratic equation is one in which the highest power to which x is raised is 2. The general quadratic may thus be represented as:

$ax^2 + bx + c$

You should already be familiar with the quadratic formula for finding roots of a quadratic polynomial.

Quadratic Formula

If $ax^2 + bx + c = 0$

Then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example

Solve $3x^2 - 5x - 7 = 0$

Answer

$3x^2 - 5x - 7 = 0$

The quadratic formula is:

If $ax^2 + bx + c = 0$

Then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Thus, here

$a = 3,\ b = -5,\ c = -7$

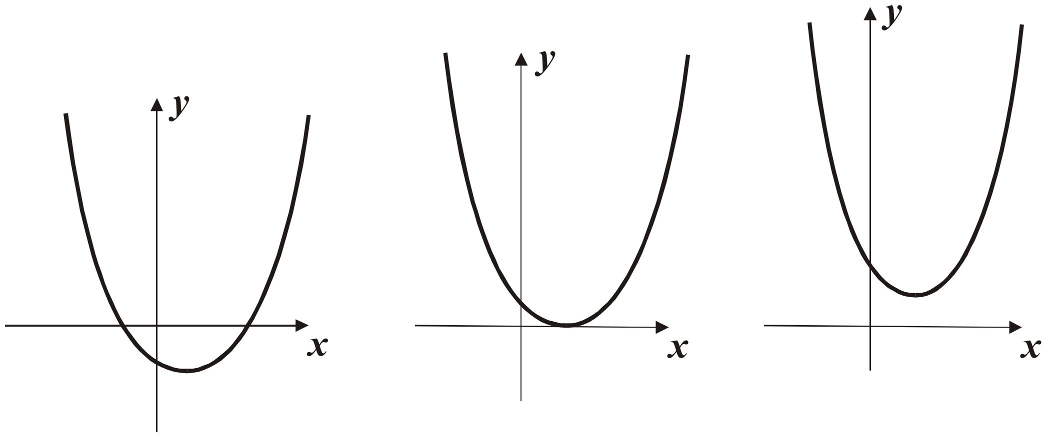
$$x = \frac{5 \pm \sqrt{(-5)^2 - (4 \times 3 \times -7)}}{6} = \frac{5 \pm \sqrt{109}}{6}$$

$$= \frac{5 + 10.44}{6} \text{ or } \frac{5 - 10.44}{6}$$

$$= 2.57 \text{ or } -0.91 \text{ (2.D.P)}$$

The roots of a quadratic are the values of x for which it is zero, and so at the roots the graph of $ax^2 + bx + c$ crosses the x-axis.

Geometric intuition indicates that quadratic may have 0, 1 (a repeated root) or 2 real roots. That is



2 roots 1 root 0 roots

(Note: Strictly speaking, all quadratics have 2 roots, but these may be "complex numbers". The subject of complex numbers is more advanced, and is introduced in another unit. A quadratic with no real roots has two complex roots.)

Discriminant

Examination of the general form of the quadratic polynomial, and its solution

If $ax^2 + bx + c = 0$

Then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Indicates that whether the quadratic has 2, 1 or 0 roots depends on the value of

$\Delta = b^2 - 4ac$

This quantity is called the "discriminant".

If the discriminant is greater than zero then there will be 2 roots; if it is exactly equal to zero then there will be just 1 root; and if it is less than zero then the expression

$\sqrt{b^2 - 4ac}$

has no solution, so there are no (real) roots.

In summary

If $\Delta = b^2 - 4ac > 0$ then $ax^2 + bx + c$ has 2 real roots
If $\Delta = b^2 - 4ac = 0$ then $ax^2 + bx + c$ has 1 real root
If $\Delta = b^2 - 4ac < 0$ then $ax^2 + bx + c$ has 0 real roots

Relationships between the roots and coefficients of a quadratic polynomial

Suppose $ax^2 + bx + c = 0$ has roots α and β; then

$$ax^2 + bx + c = 0 = (x - \alpha)(x - \beta)$$

If the quadratic is written in this factorised form, the roots are immediately obvious.
Letting the right hand side equal zero gives $x = \alpha,\ x = \beta$.
Dividing both sides by a and multiplying out the brackets on the right gives

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 = x^2 - (\alpha + \beta)x + \alpha.\beta$$

By equalting coefficients

$$\alpha + \beta = -\frac{b}{a} \quad \alpha\beta = \frac{c}{a}$$

(Note: these are called the sum and product of the roots respectively, and have significance in higher order equations)

Many questions using these principles involve the algebraic manipulation of the sum & product of the roots.

Example

If the equation $6x^2 - 12x + 16 = 0$ has roots α and β, find the equations with roots

(*i*) $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

(*ii*) $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$

Solution

(*i*) $\alpha + \beta = \dfrac{-b}{a} = \dfrac{-(-6)}{3} = 2$

$$\alpha\beta = \frac{c}{a} = \frac{8}{3}$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{2}{8/3} = \frac{3}{4}$$

This is the value of $\dfrac{-b}{a}$ for the new equation.

$$\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{1}{\beta} = \frac{3}{8}$$

This is the value of β for the new equation

Thus, the new equation is

$$x^2 - \frac{3}{4}x + \frac{3}{8} \quad or \quad 8x^2 - 6x + 3$$

(*ii*) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} = \left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)^2 - \dfrac{2}{\alpha\beta} = \left(\dfrac{3}{4}\right)^2 - 2\left(\dfrac{3}{8}\right) = \dfrac{-3}{16}$

$$\left(\frac{1}{\alpha^2}\right)\left(\frac{1}{\beta^2}\right) = \frac{1}{(\alpha\beta)^2} = \left(\frac{3}{8}\right)^2 = \frac{9}{64}$$

The new equation is

$$x^2 + \frac{3}{16}x + \frac{9}{64} = 0 \quad \text{or} \quad 64x^2 + 12x + 9 = 0$$