Roots and curve sketching

Roots

It is important to be able to sketch the graph of a polynomial function. This is an invaluable tool of problem solving as well as interesting mathematics in its own right. In this chapter our aim is to develop several techniques for sketching graphs of polynomial functions, though the whole story cannot be completed at one time, and other techniques are introduced in later chapters.

Let $y = f(x)$ be a polynomial function. The value of x that gives $f(x) = 0$ is called a *root* of the polynomial function. You should be aware that the root of a function is a particular value of the variable x; whereas the variable x represents *any* value. Therefore, it is usual to use a different symbol for a root to distinguish it from the variable. Typically, several different symbols are regularly used to denote a root of a function. For example, if $y = f(x)$ is a polynomial function we often use letters from the beginning of the alphabet, a, b, ... to denote a root of this function; the use of Greek letters $\alpha, \beta, \ldots$ is also very popular; finally we may use a variable with a subscript, $x_0, x_1, \ldots$ (these are read "x sub 0", "x sub 1" and so forth).

Using Greek letters for the present, we denote roots of $y = f(x)$ by one of $\alpha, \beta, \ldots$ and so forth. This means, if α is a root of $y = f(x)$ then $f(\alpha) = 0$. The significance of a root to the graph of a polynomial function is that at a root the graph crosses the x–axis. You should already be familiar with this idea from your work on quadratics.

Example (1)

By finding the roots of the quadratic function

$$y = x^2 + x - 1$$

sketch its graph.

Solution

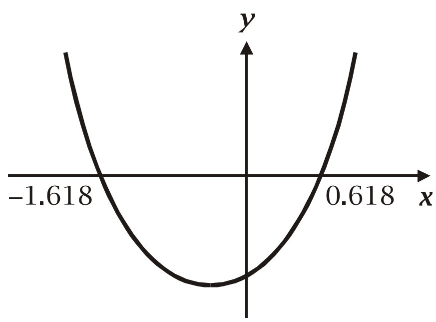
To find the roots of $x^2 + x - 1$ we shall use the quadratic formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

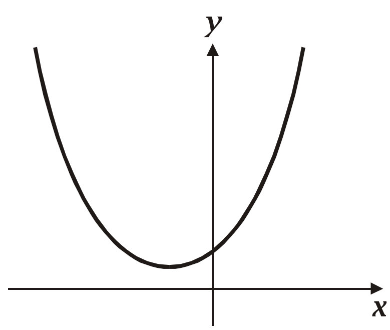
$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 + 4}}{2}$$
$$x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$
$$x_1 = 0.618 \text{ or } x_2 = -1.618 \; (3.s.f.)$$

(Note here that we have naturally used the symbols x_1, x_2 to denote the roots of this function because they come out as a pair of numbers from the use of the quadratic formula.)

The graph of $y = x^2 + x - 1$ crosses the x-axis at $x_1 = 0.618$ or $x_2 = -1.618$. Since all quadratic functions have the same basic shape – that of a parabola – we can in fact sketch this graph immediately.

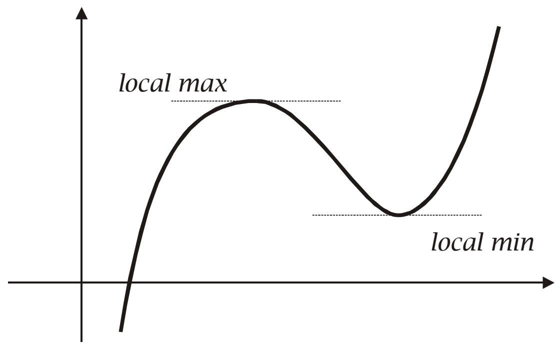


This method of looking for roots does not tell us where the minimum of the graph lies – for that we would need to use the method of completing the square. Furthermore, if the quadratic function does not have roots, then this method will not work.



A quadratic function that does not have roots.

However, this method can be extended where appropriate to assist in the sketching of other polynomial functions of higher degree. For this purpose you must also know that a polynomial of degree n can have at most $n-1$ *turning points*, where a turning point is a change of direction about a *local maximum* or *minimum*. (The proof of this assertion requires further ideas, so we have to ask you to simply accept it on trust!)



A function with a local maximum and minimum

This means that a cubic function can have *at most* two turning points and that it can have *at most* three roots and cross the *x*-axis *at most* three times. The phrase "at most" is needed because a cubic function may have less than two turning points. With this extra piece of information, we can use results about the roots of a polynomial to sketch its curve in certain cases.

Example (2)

The cubic function $f(x) = 2x^3 - 5x^2 + x + 2$ has three roots.

(a) Use the factor theorem to find one root of $f(x)$.

(b) By means of polynomial division or otherwise, completely factorise $f(x)$.

(c) Sketch the graph of $y = f(x)$.

Solution

(a)

$$f(x) = 2x^3 - 5x^2 + x + 2$$
$$f(0) = 2$$
$$f(1) = 2 - 5 + 1 + 2 = 0$$

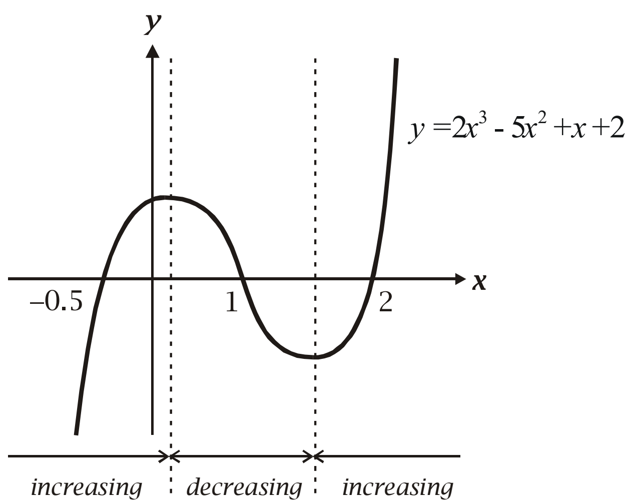
Therefore, $\alpha = 1$ is a root of $f(x)$ and $(x-1)$ is a factor.

(b)

$$\begin{array}{r|l} & 2x^2 - 3x - 2 \\ \hline x - 1 & 2x^3 - 5x^2 + x + 2 \\ & \underline{2x^3 - 2x^2} \\ & -3x^2 + x \\ & \underline{-3x^2 + 3x} \\ & -2x + 2 \\ & \underline{-2x + 2} \end{array}$$

$$f(x) = (x-1)\left(2x^2 - 3x - 2\right) = (x-1)(2x+1)(x-2)$$

(c) We know now that $f(x) = 2x^3 - 5x^2 + x + 2$ crosses the x-axis at $x = 1, x = 2$ and $x = -0.5$. Since $f(x)$ has at most two turning points it must change direction between the roots. Substitution of $x = -1$ shows that $f(-1) = -6 > 0$, so $f(x)$ is negative for $x < -0.5$. This means that $f(x)$ must be *increasing* up to the first turning point (a maximum) crossing the x-axis as it does so at $x = -0.5$, then *decreasing* down to the second turning point (a minimum), crossing the x-axis at $x = 1$; finally increasing as x continues to increase, and again crossing the x-axis at $x = 2$.



At this time we do not know the precise location of the turning points; nonetheless, knowledge of where roots lie does enable us to sketch this function.