Roots and Recurrence Relations

Prerequisites

You should be familiar with (1) the use of the differential calculus to sketch curves, (2) the use of sketches of curves to locate roots to equations; and (3) the use of the method of trial an improvement to find a numerical approximation to the root of an equation.

Example (1)

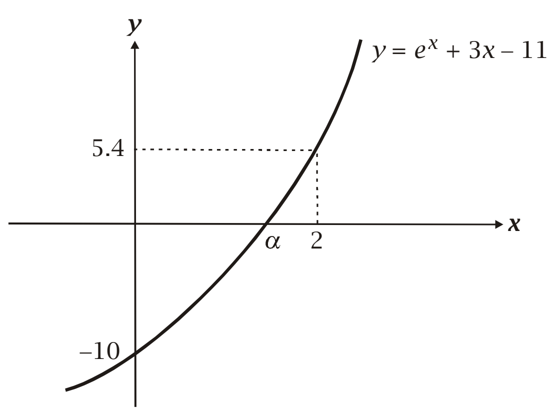
Show that the curve $y = e^x + 3x - 11$ has no a turning point and use the sketch to show that the equation $e^x + 3x - 11 = 0$ has only one root. Find the root correct to 1 decimal place using the method of trial and improvement.

Solution

$y = e^x + 3x - 11$

$\frac{dy}{dx} = e^x + 3 > 0$ for all x

For turning points $\frac{dy}{dx}$; hence since $\frac{dy}{dx} > x$ for all x, $y = e^x + 3x - 11$ has no turning points. At $x = 0$, $y = -10 < 0$. The function $y = e^x + 3x - 11$ is an monotone increasing function; that is $y \to \infty$ as $x \to \infty$. At $x = 2$ we have $y = 5.389... > 0$ so a root lies between $x = 0$ and $x = 2$.



By trial and improvement

$y(0) = -10$

$y(2) = 5.389... > 0$	$0 < \alpha < 2$
$y(1) = -5.281... < 0$	$1 < \alpha < 2$
$y(1.5) = -2.108... < 0$	$1.5 < \alpha < 2$
$y(1.7) = -0.426.. < 0$	$1.7 < \alpha < 2$
$y(1.8) = 0.449... > 0$	$1.7 < \alpha < 1.8$
$y(1.75) = 0.0046.. > 0$	$1.7 < \alpha < 1.75$

$\alpha = 1.7 \quad (\text{to 1 d.p.})$

The purpose of this chapter is to develop alternative methods to trial and improvement to the process of numerical solutions to roots of equations.

Recurrence relations

A sequence is a string of numbers in a given order. It is usual to denote successive members of the sequence by letters with numerical subscripts

$u_0,\ u_1,\ u_2,\ u_3, \ldots, u_n, \ldots$

Here u_n means the nth number in the sequence. A *recurrence relation*, also called a *recursion formula*, is a rule for generating the next member of a sequence from one or more previous members. Consider the following rule.

$u_{n+1} = u_n + 5$

This says that the $(n+1)$th number in the sequence is generated from the nth number in the sequence by adding 5 to that number. Starting with $u_0 = 1$ this generates the sequence

1 6 11 16 21 ...

The dots indicate that the sequence may be continued indefinitely. The recurrence relation may involve more than one member of the numbers already generated in the sequence. The Fibonnacci sequence, is given by the recursion formula

$u_{n+1} = u_n + u_{n-1}$

When $u_0 = 1$ and $u_1 = 1$ this gives the sequence

$u_0 = 1$

$u_1 = 1$

$u_2 = u_0 + u_1 = 1 + 1 = 2$

$u_3 = u_1 + u_2 = 1 + 2 = 3$

$u_4 = u_2 + u_3 = 2 + 3 = 5$

$u_5 = u_3 + u_4 = 3 + 5 = 8$

$u_6 = 13 \ldots$

When the magnitude of the terms in a sequence gets larger and larger, the sequence is said to diverge, or be divergent. The above sequence is divergent. If the sequence tends towards a single value, then it is said to converge, or to be convergent. Strictly proving that a sequence is convergent requires more advanced ideas, but here we will use the informal argument that if a sequence is obviously converging then it is converging. In questions at this level it will always be stated whether the sequence generated by a recurrence relation converges or not, or convergence will be implicit in the question.

Example (3)

A recurrence relation is given by

$u_{n+1} = \ln(u_n + 5)$

Starting with the value $u_0 = 1.7$ find the values of u_1, u_2, u_3, u_4, u_5 to 5 decimal places and state whether the sequence generated is convergent or divergent.

Solution

$u_{n+1} = \ln(u_n + 5)$

$u_{n+1} = \ln(u_n + 5)$

$u_0 = 1.7$

$u_1 = \ln(6.7) = 1.90211$

$u_2 = 1.93183$

$u_3 = 1.93612$

$u_4 = 1.93674$

$u_5 = 1.93683$

The difference between successive terms is getting smaller and smaller, so intuitively this sequence is convergent.

Roots and recurrence relations

When a recurrence relation generates a convergent sequence, then the sequence "homes in" upon a given number. The sequence

$u_0, \; u_1, \; u_2, \; u_3, \; . \; . \; . \; . \; , u_n, \; . \; . \; . \; .$

gets closer and closer to some number α the larger n gets. It is an approximation to the real number α, and the further we go the better the approximation is. Therefore, recurrence relations can be used to generate numerical approximations to roots of equations.

This is useful because the method of trial and improvement is *inefficient*. It involves a lot of work and a good approximation takes a lot of repeats of the process. In the previous example the recurrence formula

$u_{n+1} = \ln(u_n + 5)$

rapidly generated a convergent sequence. The last two terms of the sequence we generated

$u_4 = 1.93674$

$u_5 = 1.93683$

are equal to 3 decimal places.

$u_4 = 1.937 = u_5$ (3 d.p.)

In questions at this level you will be given a recurrence relation and told that the relation converges on the root of an equation. (At a higher level you may learn how to derive recurrence relations that converge on the roots of equations.)

Example (4)

(*a*) Show that the equation

$e^{3a} - a - 8 = 0$

has a root α between 0.5 and 1.

(*b*) The recurrence relation

$a_{n+1} = \frac{1}{3}\ln(a_n + 8)$

with $a_0 = 1.7$ can be used to find α. Find and record the values of a_1, a_2, a_3, a_4 giving your answers to 7 decimal places. Write down the value of a_4 to 5 decimal places and prove that this value is the value of α correct to five decimal places.

Solution

(*a*) Let $f(a) = e^{3a} - a - 8$ then

$f(0.5) = e^{1.5} - 0.5 - 8 = -4.018...$

$f(1) = e^3 - 1 - 8 = 11.085...$

There is a sign change between $a = 0.5$ and $a = 1$ therefore a root α lies in this interval.

(*b*) $a_{n+1} = \frac{1}{3}\ln(a_n + 8)$

$a_0 = 0.7$

$a_1 = 0.7211077$

$a_2 = 0.7219154$

$a_3 = 0.7219463$

$a_4 = 0.7219475$

To 5 decimal places $a_4 = 0.72195$

Since to 5 decimal places $a_4 = a_3 = 0.72195$ we have $\alpha = 0.72195$.