Roots of Polynomials of Degree 3

Complex conjugates as roots of quadratics with negative discriminant

You should be familiar with the solution to the quadratic equation $ax^2 + bx + c = 0$ and its solution by complex numbers. The quadratic formula is

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The term $\Delta = b^2 - 4ac$ is called the discriminant. When $\Delta < 0$ the quadratic equation does not have real roots. Let z denote a complex number. If $z = x + iy$ then $\bar{z} = x - iy$ is called the *conjugate* of *z*. When $\Delta < 0$ the quadratic equation has two solutions

$z = x + iy$ and $\bar{z} = x - iy$

that are complex conjugates of each other.

Example (1)

Find the roots of $x^2 - 4x + 7 = 0$ expressing your answer as complex numbers.

Solution

Substitution into the quadratic formula gives

$$\begin{aligned} x &= \frac{4 \pm \sqrt{16 - 28}}{2} \\ &= \frac{4 \pm \sqrt{-12}}{2} = \frac{4 \pm 2\sqrt{-3}}{2} \\ &= 2 + \sqrt{3}i \text{ or } 2 - \sqrt{3}i \end{aligned}$$

In this example we see that the two solutions of the polynomial are complex conjugates of each other.

Relationships between roots of quadratic equations

You should already be familiar with the properties of roots of quadratic equations. To remind you, if $ax^2 + bx + c = 0 = a(x - \alpha)(x - \beta)$ is a quadratic equation with roots α and β then

$$\alpha + \beta = -\frac{b}{a} \quad \alpha\beta = \frac{c}{a}$$

Example (2)

Given that α and β are the roots of the equation $4x^2+3x+1=0$ find an equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.

Solution

$$4x^2+3x+1=0$$

$$x^2+\frac{3}{4}x+\frac{1}{4}=0$$

$$(x-\alpha)(x-\beta)=0$$

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

$$\alpha+\beta=-\frac{3}{4}$$

$$\alpha\beta=\frac{1}{4}$$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{9}{16}-\frac{1}{2}=\frac{1}{16}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}=\frac{1/16}{1/16}=1$$

$$\left(x-\frac{1}{\alpha^2}\right)\left(x-\frac{1}{\beta^2}\right)=x^2-\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)x+\frac{1}{(\alpha\beta)^2}=0$$

So the equation is

$$x^2-x+16=0$$

Roots and linear factors

A polynomial is a function of the form $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ where $a_n, a_{n-1}, \ldots a_1, a_0$ are constants called *coefficients*, and $a_n\neq 0$. The *degree* of the polynomial is the highest power of x in the expression $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$. First we state that any polynomial with distinct roots can be written as a product of linear factors.

Theorem

If $f(x)$ is a polynomial of defree $n\geq 1$ with n roots $\alpha_1,\ \alpha_2, \ldots.\ \alpha_n$ then

$$f(x)=c(x-\alpha_1)(x-\alpha_2)\ldots..(x-\alpha_n)$$

where c is a non-zero constant. This theorem can be proven by mathematical induction.

The terms $(x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)$ are called the *factors* of the polynomial. The result just referred to shows the following.

$f(x)$ is a polynomial with $n \geq 1$ factors $(x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)$

$$\Downarrow$$

$f(x)=(x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)$ is a polynomial of degree *n*.

The converse of this is

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $n \geq 1$ is a polynomial of degree n

$$\Downarrow$$

$f(x)$ has n factors $(x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)$ and $f(x)=(x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)$

This converse statement is called *the fundamental theorem of algebra* and its proof is beyond the scope of this text. However, what the two theorems say is that any polynomial of degree n can be factorised into n factors (that is, has n roots). Here also we must allow the roots, α_1, α_2, ... α_n, to be include complex roots, for if the roots are only real numbers then this converse does not hold in general. The two theorems just stated refer to factors $(x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)$ but for the fundamental theorem to be valid it must also be understood that the factors may be *repeated.* We explain below what this means.

Repeated roots

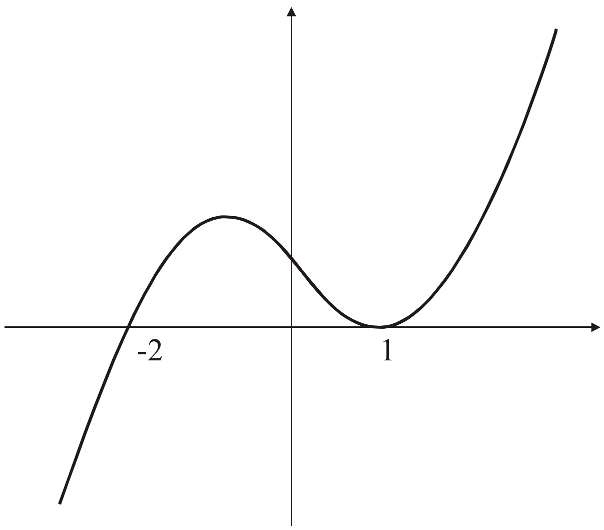
To illustrate what is meant by *repeated* roots, consider the function

$f(x) = x^3 - 3x + 2 = (x-1)^2(x+2)$

The root $(x-1)$ occurs twice in

$f(x) = x^3 - 3x + 2 = (x-1)^2(x+2)$

When a root occurs two or more times in the factorised form of the function it is called a *repeated root.* The graph that follows illustrates that $f'(x) = 0$ at the root $x = 1$.



That is, if $(x-\alpha)$ is a repeated root of $f(x)=a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ then $f'(\alpha)=0$. To prove this generally, let

$$f(x)=(x-\alpha)^2 g(x)$$

be a function with a repeated root, then

$$\begin{aligned} f'(x) &= 2(x-\alpha)g(x)+(x-\alpha)^2 g'(x) \\ &= (x-\alpha)\left(2g(x)+(x-\alpha)g'(x)\right) \end{aligned}$$

Hence

$$f'(\alpha)=0$$

So the criterion to test whether a function has a repeated root at α is $f'(\alpha)=0$. The factor theorem can be used to find the roots of a polynomial function and then the condition

$f'(\alpha)=0 \Leftrightarrow (x-\alpha)$ is a repeated root

can be used to test for repeated roots if necessary.

Cubic equations

A polynomial function of degree 3 is also called a *cubic*. A *cubic equation* is an equation of the form $ax^3+bx^2+cx+d=0$. You are expected to be able to apply your knowledge of the factors of cubics to solve problems involving cubic equations. This basically involves writing the cubic equation as a product of its *three* factors. If α, β, γ are the roots of a cubic equation, then

$$(x-\alpha)(x-\beta)(x-\gamma)=0$$

Example (3)

The roots of the cubic equation

$$x^3 + px^2 + \frac{63}{8}x + q = 0$$

Form a geometric series with common ratio $1/2$. Given that the first three terms of these series are all positive, find (a) all three roots, and (b) the values of p and q.

Solution

The first three terms of a geometric series are

$$a,\ ar,\ ar^2$$

where a is the first term and r is the common ration. Here $r = 1/2$ so the factors of this equation are

$$(x-a)\left(x-\frac{a}{2}\right)\left(x-\frac{a}{4}\right)$$

That is

$$\begin{aligned} x^3 + px^2 + \frac{63}{8}x + q &\equiv (x-a)\left(x-\frac{a}{2}\right)\left(x-\frac{a}{4}\right) \\ &\equiv (x-a)\left(x^2 - \frac{3}{4}ax + \frac{a^2}{8}\right) \\ &\equiv x^3 - \frac{7}{4}ax^2 + \frac{7}{8}a^2x - \frac{a^3}{8} \end{aligned}$$

Equating coefficients gives

$$p = -\frac{7}{4}a$$

$$\frac{63}{8} = \frac{7}{8}a^2$$

$$q = -\frac{9}{8}a^3$$

Solving these equations gives

$a^3 = 9$ $\qquad$ $a = 3$ $\qquad$ [Must be the positive root]

$p = -\frac{21}{4}$ $\qquad$ $q = -\frac{27}{8}$

And the roots are

$$(x-a)\left(x-\frac{a}{2}\right)\left(x-\frac{a}{4}\right) = (x-3)\left(x-\frac{3}{2}\right)\left(x-\frac{3}{4}\right)$$

Relationships between the roots of cubic equations

If α, β, γ are the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Proof

Let $ax^3 + bx^2 + cx + d = 0$ be a cubic equation with roots α, β, γ, then

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

then

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} \equiv (x - \alpha)(x - \beta)(x - \gamma)$$

$$\equiv x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$$

By equating coefficients we obtain

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

The roots, α, β, γ cannot be found directly by means of these equations. Any attempt to solve them simultaneously merely regenerates the original equation in some form. However, if additional information is available, then these equations may provide a shortcut to the roots. We may use the symbols

$\sum\alpha$ to denote $\alpha + \beta + \gamma$

$\sum\alpha\beta$ to denote $\alpha\beta + \beta\gamma + \gamma\alpha$

that is

$$\sum\alpha = \alpha + \beta + \gamma$$

$$\sum\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha$$

Using these symbols we can prove a number of identities.

(1) $\left(\sum\alpha\right)^2 = \sum\alpha^2 + 2\sum\alpha\beta$

Proof

$$
\begin{aligned}
\text{LHS} &= \left(\sum \alpha\right)^2 \\
&= (\alpha+\beta+\gamma)^2 \\
&= (\alpha+\beta+\gamma)(\alpha+\beta+\gamma) \\
&= \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma) \\
&= \sum \alpha^2 + 2\sum \alpha\beta \\
&= \text{RHS}
\end{aligned}
$$

(2) $\alpha\beta\gamma\sum\alpha^{-1} = \sum\alpha\beta$

Proof

$$
\begin{aligned}
\text{LHS} &= \alpha\beta\gamma\sum\alpha^{-1} \\
&= \alpha\beta\gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right) \\
&= \beta\gamma+\alpha\gamma+\alpha\beta \\
&= \sum\alpha\beta \\
&= \text{RHS}
\end{aligned}
$$

(3) $\sum\alpha^3 - 3\alpha\beta\gamma = \left(\sum\alpha\right)\left(\sum\alpha^2-\sum\alpha\beta\right)$

Proof

$$
\begin{aligned}
\text{RHS} &= \left(\sum\alpha\right)\left(\sum\alpha^2-\sum\alpha\beta\right) \\
&= (\alpha+\beta+\gamma)\left(\alpha^2+\beta^2+\gamma^2-\alpha\beta+\beta\gamma+\gamma\alpha\right) \\
&= \alpha^3+\beta^3+\gamma^3+\alpha\beta^2+\alpha\gamma^2+\beta\alpha^2+\beta\gamma^2+\gamma\alpha^2+\gamma\beta^2 \\
&\qquad -\alpha\beta^2-\alpha^2\gamma-\alpha\beta\gamma-\beta^2\alpha-\alpha\beta\gamma+\beta^2\gamma-\alpha\beta\gamma-\gamma^2\alpha+\gamma^2\beta \\
&= \alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma \\
&= \text{LHS}
\end{aligned}
$$

Example (4)

The roots of the cubic equation

$2x^3 - 5x^2 - 4x + 3 = 0$

are denoted by α, β, γ. Find the cubic equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

Solution

From $2x^3 - 5x^2 - 4x + 3 = 0$ and by substitution into

$$\alpha+\beta+\gamma=-\frac{b}{a}$$
$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$$
$$\alpha\beta\gamma=-\frac{d}{a}$$

we have

$$\alpha+\beta+\gamma=\frac{5}{2}$$
$$\alpha\beta+\beta\gamma+\gamma\alpha=-2$$
$$\alpha\beta\gamma=-\frac{3}{2}$$

The cubic whose roots are $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ can be factorised as

$$\begin{aligned}
g(x)&=\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\left(x-\frac{1}{\gamma}\right)\\
&=\left(x-\frac{1}{\alpha}\right)\left(x^2-\left(\frac{1}{\beta}+\frac{1}{\gamma}\right)x+\frac{1}{\beta\gamma}\right)\\
&=x^3-x^2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)+x\left(\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}\right)-\frac{1}{\alpha\beta\gamma}\\
&=x^3-x^2\left(\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}\right)+x\left(\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}\right)-\frac{1}{\alpha\beta\gamma}\\
&=x^3-x^2\left(\frac{-2}{-3/2}\right)+x\left(\frac{5/2}{-3/2}\right)-\left(\frac{1}{-3/2}\right)\\
&=x^3-\frac{4}{3}x^2-\frac{5}{3}x+\frac{2}{3}
\end{aligned}$$

The required equation is

$x^3-\frac{4}{3}x^2-\frac{5}{3}x+\frac{2}{3}=0$ or

$3x^3-4x^2-5x+2=0$

Example (5)

A cubic equation $ax^3+bx^2+cx+d=0$ has roots $\alpha,\ \beta,\ \gamma$. Find a cubic with roots

$3\alpha-5,\ 3\beta-5,\ 3\gamma-5$

Solution

To illustrate the meaning of this question, the function

$f(x)=ax^3+bx^2+cx+d$

is effectively subject to a scaling by a factor of 3 and a translation by a factor of +5, Let $y = 3x - 5$

Then $x = \frac{y+5}{3}$

The roots of $ax^3 + bx^2 + cx + d = 0$ will be $\alpha,\ \beta,\ \gamma$ if, and only if $3\alpha - 5,\ 3\beta - 5,\ 3\gamma - 5$ are the roots of

$$a\left(\frac{y+5}{3}\right)^3 + b\left(\frac{y+5}{3}\right)^2 + c\left(\frac{y+5}{3}\right) + d = 0$$
$$\frac{a}{27}\left(y^3 + 15y^2 + 75y + 125\right) + \frac{b}{9}\left(y^2 + 10y + 25\right) + \frac{c}{3}\left(y+5\right) + d = 0$$
$$a\left(y^3 + 15y^2 + 75y + 125\right) + 3b\left(y^2 + 10y + 25\right) + 9c\left(y+5\right) + 27d = 0$$
$$ay^3 + y^2\left(15a + 3b\right) + y\left(75a + 30b + 9c\right) + \left(125a + 75b + 45c + 27d\right) = 0$$

The following graph illustrates this problem