Scalar Fields and Vector Functions

Scalar Field

For example, let (x, y) denote a position in two-dimensional space and let $P(x, y)$ represent the pressure at that point.

The variable P is one-dimensional quantity and is a function of the two variables x and y. That is

$$P = P(x, y)$$

It is an example of a scalar field. Since it is a function of two variables, it is an example of a two dimensional scalar field.

A three dimensional scalar field would be a function of three variables and an n dimensional scalar field is a function of n variables. In general an n-dimensional scalar field is a function.

$$\phi \begin{cases} \mathbb{R}^n \to \mathbb{R} \\ (x_1, x_2, ..x_n) \to \phi(x_1, x_2 ... x_n) \end{cases}$$

Practical examples of scalar fields in physics are the assignment of a temperature to points of a body, and the pressure of the air of the Earth's atmosphere. These are both mappings from a three dimensional space to a single real number, and so are examples of three dimensional scalar fields.

<u>Example</u>

The Euclidean distance from a point

$$\mathbf{r} = (x, y, z)$$

from a fixed point

$$\mathbf{p} = (x_0, y_0, z_0)$$

is an example of a scalar field. It is given by the formula

$$f(\mathbf{r}) = f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

Contour Curves

Suppose we have a two-dimensional scalar field $\phi\ (x, y)$. Then a curve will be defined by each specific value that $\phi\ (x, y)$ can take. The curve $\phi\ (x, y) = k$ is called a contour curve of the scalar field.

(Note that the contour curves of a temperature field are called isotherms and the contour curves of a pressure field are called isobars.)

Example

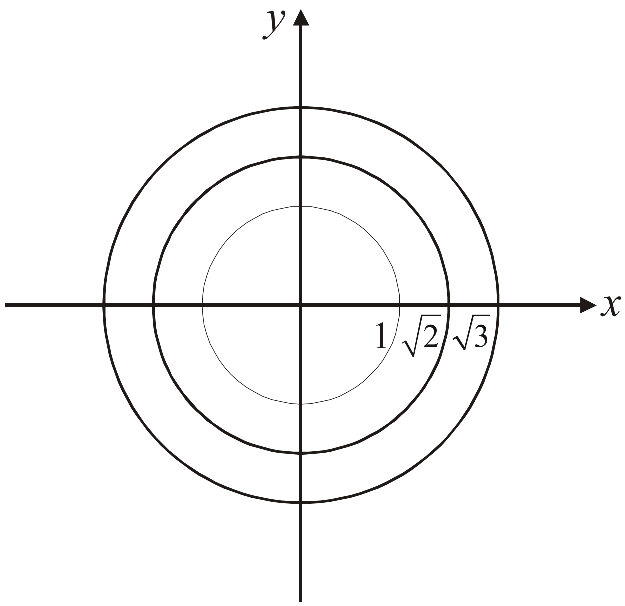
Let
$\mathbb{R}^2 \to \mathbb{R}$
$\phi\ (x, y) = x^2 + y^2$
be a scalar field

Sketch the contours given by

$\phi(x, y) = 1$
$\phi(x, y) = 2$
$\phi(x, y) = 3$

Solution

We have $\phi(x, y) = 1$
that is $x^2 + y^2 = 1$
This is the equation of the circle with centre the origin 4 radius1. Similary
$\phi(x, y) = 2$ is a circle with radius $\sqrt{2}$ and $\phi(x, y) = 3$ is a circle with radius $\sqrt{3}$.



A contour map is a map of a surface given by its contours.

If we make a contour map by taking a constant interval between each contour; e.g.

$$\phi(x, y) = 10$$
$$\phi(x, y) = 20$$
$$\phi(x, y) = 30$$

where the difference between successive contours is always 10 units, then the rate of increase or decrease of a scalar field is related to the closeness of the contour curves. The closer the contour curves are together the faster the scalar field is changing.

Contour Surfaces

The idea of a contour curve can be generalised to 3-dimensions.

$$\phi(x, y, z) = k$$

This will give pictorially a series of contour surfaces. For example, if $\phi(x, y, z) = x^2 + y^2 + z^2$ then the contour surfaces given by $x^2 + y^2 + z^2 = k$ for different values of k are a series of rested spheres

.A contour surface can be defined for a n-dimensional scalar field

$$\phi(x, x_2 ... x_n)$$

but it is not possible to visualise this in 3-dimensional space.

Vector functions

We may imagine a particle moving along a contour curve; for example, along the contour curve

$$x^2 + y^2 = 1$$

of the scalar field $\phi\ (x, y) = x^2 + y^2$. This movement may be a function of time, or some other parameter. More general the position of a particle in space may be given by a vector function

$$\mathbf{v}(t) = \left(v_1(t), v_2(t), v_3(t)\right)$$

where t is a parameter. This is a vector function of 3 dimensions, but clearly the concept could be applied to 2 dimensions or spaces of dimension greater than 3. Such a vector function may not be always continuous or differentiable, but if it is, then its derivative will be defined in the usual way, by

$$\frac{d}{dt}\mathbf{v}(t) = \left(\frac{d}{dt}v_1(t), \frac{d}{dt}v_2(t), \frac{d}{dt}v_3(t)\right) \qquad \mathbf{v}'(t) = \left(v_1'(t), v_2'(t), v_3'(t)\right)$$

That is, by differentiating each of the components of the vector function.

The usual rules for differentiating scalar multiples and sum of functions applies to vector functions. Specifically

$$\frac{d}{dt}(c\mathbf{v}) = c\frac{d}{dt}\mathbf{v} \qquad (c\mathbf{v})' = c\mathbf{v}'$$

$$\frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d}{dt}\mathbf{u} + \frac{d}{dt}\mathbf{v} \qquad (\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

Differentiation of scalar and vector products.

Suppose $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ is a vector field

The functions $f_{1,} f_2, f_3$ are its Cartesian coordinates this function. Then the vector field can be differentiated according to the obvious rule.

$$\frac{d\mathbf{F}}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{df_1}{dt}\mathbf{j} + \frac{dt_3}{dt}\mathbf{k}$$

Then differentiation of scalar and vector (cross) products of vectors follows the normal product (Leibniz) rule.

$$\frac{d}{dt}(\mathbf{F}\cdot\mathbf{G}) = \frac{d\mathbf{F}}{dt}\mathbf{G} + \mathbf{F}\frac{d\mathbf{G}}{dt}$$

$$\frac{d}{dt}(\mathbf{F}\times\mathbf{G}) = \frac{d\mathbf{F}}{dt}\times\mathbf{G} + \mathbf{F}\times\frac{d\mathbf{G}}{dt}$$

We will prove the result for the cross product. That is

$$\frac{d}{dt}(\mathbf{F}\times\mathbf{G}) = \frac{d\mathbf{F}}{dt}\times\mathbf{G} + \mathbf{F}\times\frac{d\mathbf{G}}{dt} \qquad (*)$$

Let

$\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ *and* $\mathbf{G} = g_1\mathbf{i} + g_2\mathbf{j} + g_2\mathbf{k}$

then the left-hand-side of $(*)$ is

$$\begin{aligned}\text{LHS} &= \frac{d}{dt}(\underline{\mathbf{F}\times\mathbf{G}}) \\ &= \frac{d}{dt}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3\end{vmatrix}\end{aligned}$$

$$= \begin{vmatrix} f_2 & f_3 \\ g_2 & g_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} f_3 & f_1 \\ g_3 & g_1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}$$

$$= \frac{d}{dt}\left\{ \left(f_2 g_3 - f_3 g_2 \right) \mathbf{i} + \left(f_3 g_1 - f_1 g_3 \right) \mathbf{j} + \left(f_1 g_2 - f_2 g_1 \right) \mathbf{k} \right\}$$

$$= \left(\frac{d}{dt}\left(f_2 g_3 \right) - \frac{d}{dt}\left(f_3 g_2 \right) \right) \mathbf{i} + \left(\frac{d}{dt}\left(f_3 g_1 \right) - \frac{d}{dt}\left(f_1 g_3 \right) \right) \mathbf{j} + \left(\frac{d}{dt}\left(f_1 g_2 - f_2 g_1 \right) \right) \mathbf{k}$$

$$= \left(f_2' g_3 + f_2 g_3' - f_3' g_2 - f_3' g_2 \right) \mathbf{i} + \left(f_3' g_1 + f_3 g_1' - f_1' g_3 - f_1 g_3' \right) \mathbf{j}$$

$$+ \left(f_1' g_2 + f_1 g_2' - f_2' g_1 - f_2 g_1' \right) \mathbf{k}$$

However the right-hand side of (*) is

$$\text{RHS} = \frac{d\mathbf{F}}{dt} \times \mathbf{G} + \mathbf{F} \times \frac{d\mathbf{G}}{dt}$$

$$= \left(f_1' \mathbf{i} + f_2' \mathbf{j} + f_3' \mathbf{k} \right) \times \left(g_1 \mathbf{i} + g_2 \mathbf{j} + g_3 \mathbf{k} \right) + \left(f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k} \right) \times \left(g_1' \mathbf{i} + g_2' \mathbf{j} + g_3' \mathbf{k} \right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1' & g_2' & g_3' \end{vmatrix}$$

$$= \begin{vmatrix} f_2' & f_3' \\ g_2 & g_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} f_3' & f_1' \\ g_3 & g_1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} f_1' & f_2' \\ g_1 & g_2 \end{vmatrix} \mathbf{k} + \begin{vmatrix} f_2 & f_3 \\ g_2' & g_3' \end{vmatrix} \mathbf{i} + \begin{vmatrix} f_3 & f_1 \\ g_3' & g_1' \end{vmatrix} \mathbf{j} + \begin{vmatrix} f_1 & f_2 \\ g_1' & g_2' \end{vmatrix} \mathbf{k}$$

$$= \left(f_2' g_3 - f_3' g_2 \right) \mathbf{i} + \left(f_3' g_1 - f_1' g_3 \right) \mathbf{j} + \left(f_1' g_2 - f_2 g_1 \right) \mathbf{k}$$

$$+ \left(f_2 g_3' - f_3 g_2' \right) \mathbf{i} + \left(f_3 g_1' - f_1 g_3' \right) \mathbf{j} + \left(f_1 g_2' - f_2 g_1' \right) \mathbf{k}$$

$$= \text{LHS}$$