The Scalar Product of Vectors

Prerequisites

You should familiar with the vector equation of the straight line.

Example (1)

Two lines l_1 and l_2 are given by vector equations

l_1 $\quad \mathbf{r} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

l_2 $\quad \mathbf{s} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Show that these two lines are perpendicular.

Solution

The direction of the two lines l_1 and l_2 are given by the vectors $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ respectively.

The vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ represents a displacement along the line l_1 and the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ represents a displacement along the line l_2. That is, for l_1

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

where Δx is a change in the x direction and Δy is the corresponding change in the y direction. So the gradient of l_1 is $m_1 = \frac{\Delta y}{\Delta x} = -\frac{1}{2}$. For l_2 the gradient is $m_2 = \frac{\Delta y}{\Delta x} = \frac{2}{1} = 2$.

Since $m_1 \times m_2 = -\frac{1}{2} \times 2 = -1$ the two lines are perpendicular.

Note that in this example the actual position of either line in space was not needed. We only needed the direction of the lines to determine whether they were perpendicular. We did not need to know the location of any point on them. In three dimensions two lines do not necessarily intersect. However, even if two lines do not meet at a point of intersection they may still be perpendicular to one another.

The scalar product of two vectors

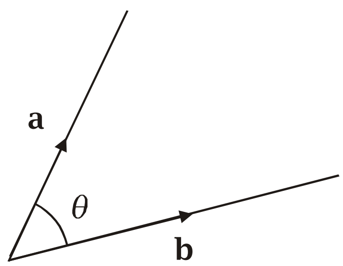
Given two vectors it is often desirable to find the angle between them. It becomes possible to do this owing to the definition of a way of multiplying two vectors together that is called the *scalar product* of two vectors. The scalar product is also commonly known as the *dot product*.

Definition of the scalar product

The scalar (dot) product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined to be:

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.



In words: the scalar (dot) product of two vectors **a** and **b** is the modulus (size) of **a** multiplied by the modulus (size) of **b** multiplied by the cosine of the angle between them. We remark that this definition is still valid even if the two vectors do not intersect. This is called the scalar product because the result of multiplying the two vectors is not another vector but rather a number, or scalar.

Example (2)

(*a*) Rearrange $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ to make $\cos\theta$ the subject.

(*b*) The scalar product of two unitary vectors is 0.5. What is the acute angle between these two vectors?

Solution

(*a*) $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

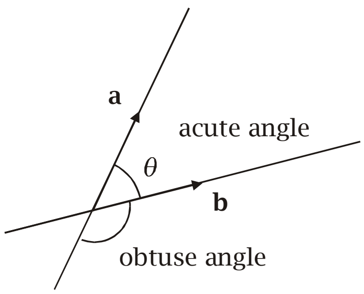
(*b*) The length (modulus) of any unitary vector is 1. Therefore, from part (*a*) the cosine of the angle between these two vectors is

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{0.5}{1\times 1} = 0.5$$

Hence the acute angle between the vectors is

$$\theta = \cos^{-1}(0.5) = 60°$$

The mention of "acute angle" in the question is relevant. There are two angles between any two vectors.



Likewise, the equation $\theta = \cos^{-1}(x)$ has two solutions in the interval $0° \le \theta \le 360$. The scalar product $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ is understood to give one of the angles between the two vectors **a** and **b** and you may be required to find the other.

Example (2) illustrates the usefulness of the scalar product. If for two vectors **a** and **b** we know $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ then we can find the angle between them. Fortunately, we can compute $\mathbf{a}\cdot\mathbf{b}$ from the component form of **a** and **b**.

Component form of the scalar product

Let $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$

and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

then $\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$

In words $\mathbf{a}\cdot\mathbf{b}$ is found by multiplying the x coordinates, the y coordinates and the z coordinates of each vector and summing these. Shortly, we shall prove this formula, but first we shall illustrate its use.

Example (3)

Find the acute angle between $\mathbf{a} = (1,2,-2)$ and $\mathbf{b} = (2,-1,2)$.

Solution

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$= \frac{(1,2,-2)\cdot(2,-1,2)}{|(1,2,-2)||(2,-1,2)|}$$

$$= \frac{2-2-4}{\sqrt{1+4+4}\cdot\sqrt{1+4+4}}$$

$$= -\frac{4}{9}$$

Therefore

$\theta = \cos^{-1}\left(-4/9\right) = 116.4°$ (nearest 0.1°)

$\therefore$ acute angle $= 180 - 116.4° = 63.6°$ (nearest 0.1°)

We remark that whilst vectors may be given in row, column or **i**, **j**, **k** form, it is easiest to find the scalar product when they are in row form.

We will now prove that the definition of the scalar product as

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

is equivalent to the result

$\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$

Theorem

Let $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$

$\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

then $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

implies $\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$

Conversely $\mathbf{a}\cdot\mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$

implies $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

Proof

The proof is in two parts, the forward derivation and the reverse derivation.

<u>Forward Derivation</u>

Let $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$

$\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$$\mathbf{a} \cdot \mathbf{b} = (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \cdot (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})$$

$$= x_1x_2\mathbf{i}\cdot\mathbf{i} + x_1y_2\mathbf{i}\cdot\mathbf{j} + x_1z_2\mathbf{i}\cdot\mathbf{k} + y_1x_2\mathbf{j}\cdot\mathbf{i} + y_1y_2\mathbf{j}\cdot\mathbf{j} + y_1z_2\mathbf{j}\cdot\mathbf{k} + z_1x_2\mathbf{k}\cdot\mathbf{i} + z_1y_2\mathbf{k}\cdot\mathbf{j} + z_1z_2\mathbf{k}\cdot\mathbf{k} \qquad (*)$$

But

$\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}||\mathbf{i}|\cos\theta$

In this $\theta = 0°$ since **i** is identical to itself, so $\cos\theta = 1$. Furthermore, $|\mathbf{i}| = 1$.

$\therefore \mathbf{i} \cdot \mathbf{i} = 1$.

Likewise

$\mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{k} \cdot \mathbf{k} = 1$.

Now $\mathbf{i} \cdot \mathbf{j} = |\mathbf{i}||\mathbf{j}|\cos\theta$

But **i** and **j** are defined to be perpendicular.

Hence in this expression, $\theta = 90°$ and $\cos\theta = 0$ and $\mathbf{i} \cdot \mathbf{j} = 0$.

Likewise, $\mathbf{i} \cdot \mathbf{k} = 0$ and any pair of different basis vectors **i**, **j**, **k** gives 0 scalar product.

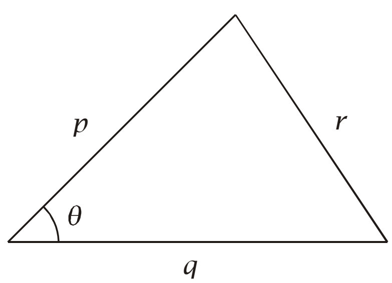
Hence, on substituting into $(*)$ we have

$\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$

<u>Reverse derivation</u>

Let $\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2$

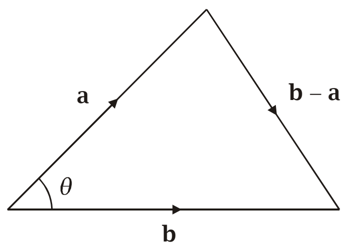
In the following diagram



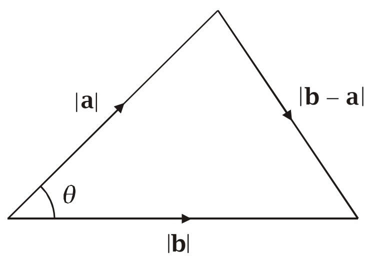
The cosine formula is $r^2 = p^2 + q^2 - 2pq\cos\theta$. Rearrangement of this gives

$$\cos\theta = \frac{p^2 + q^2 - r^2}{2pq}$$

The vectors **a**, **b** and **b** – **a** form a triangle



The lengths of this triangle can be substituted into the cosine formula



$$\cos\theta = \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - \left(|\mathbf{b}-\mathbf{a}|\right)^2}{2|\mathbf{a}||\mathbf{b}|} \qquad (*)$$

Now $|\mathbf{a}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $|\mathbf{b}| = \sqrt{x_2^2 + y_2^2 + z_2^2}$

Also $\mathbf{b} - \mathbf{a} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$

So $|\ \mathbf{b} - \mathbf{a}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Substitution into the cosine formula at $(*)$ produces, after some algebra

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}}$$

But

$$\frac{\mathbf{a}\cdot\mathbf{b}}{2|\mathbf{a}||\mathbf{b}|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}}$$

Hence

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$\therefore\ \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

Parallel and perpendicular Vectors

The scalar (dot) product can be used to test whether two vectors are parallel of perpendicular.

If two vectors are parallel, then the angle, θ, between them is 0. Hence $\cos\theta = 1$.

If two vectors are perpendicular, then the angle between them, $\theta = 90°$. Hence $\cos\theta = 0$.

Therefore

If $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$ then $\mathbf{a}$ and $\mathbf{b}$ are parallel.

If $\mathbf{a}\cdot\mathbf{b} = 0$ then $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

In fact, if **a** and **b** are parallel, then one is a scalar multiple of the other.

$\mathbf{a} = \lambda\mathbf{b}$

for some number λ. This means that the definition of the scalar product is not normally used to test whether two vectors are parallel, because it is usually obvious when one is the scalar multiple of the other. However, the dot product is used to determine whether two vectors are perpendicular. Vectors that are not perpendicular or parallel are called *skew*. Consequently, vectors are skew if

(i) $\quad \mathbf{a}\cdot\mathbf{b} \neq 0$

(ii) $\quad \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \neq 1$

Example (5)

Four vectors are given by

$\mathbf{a} = 6\mathbf{i} + \mathbf{j} - 3\mathbf{k}$
$\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$
$\mathbf{c} = -6\mathbf{i} - \mathbf{j} + 3\mathbf{k}$
$\mathbf{d} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

Find the smallest angle between the vector **a** and each of the other vectors.

Solution

$\mathbf{a}\cdot\mathbf{b} = (6,1,-3)\cdot(2,-3,3) = 12 - 3 - 9 = 0$

a and **b** are perpendicular and the angle between them is 90°.

$\mathbf{a} = -\mathbf{c}$

a and **c** are parallel and the angle between them is 180°.

$\mathbf{a}\cdot\mathbf{d} = (6,1,-3)\cdot(-2,1,3) = -12 + 1 - 9 = -20$

These vectors are skew. The angle between them is

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{d}}{|\mathbf{a}|\times|\mathbf{d}|} = \frac{-20}{\sqrt{46}\times\sqrt{14}} \qquad \Rightarrow \qquad \theta = 142.0° \quad (\text{nearest } 0.1°)$$

Acute angle given by $\quad \theta' = 180 - 142.0° = 38.0° \quad (\text{nearest } 0.1°)$

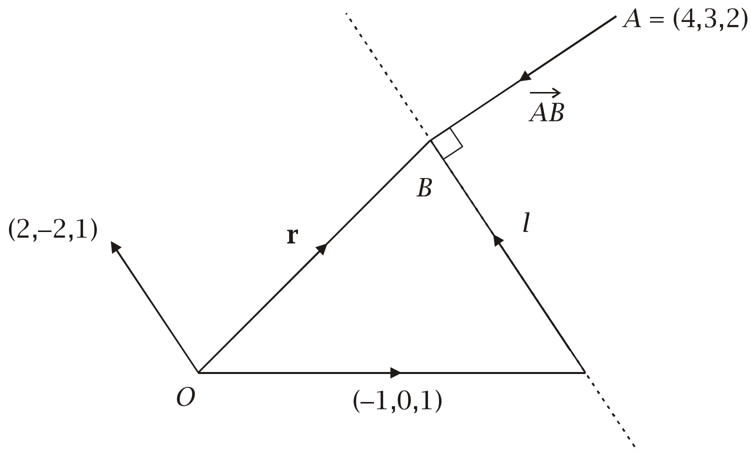
Example (6)

Find the unit vector passing through $A = (4,3,2)$ that is perpendicular to the line given by

$$\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Solution

Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the position vector of the point B on the line l such that AB is perpendicular to l.



Since $\mathbf{r}$ lies on l we have

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Then

$x = -1 + 2t$

$y = -t$

$z = 1 + 2t$

Now $A = (4,3,2)$

$$\overrightarrow{AB} = \mathbf{r} - \mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

Substituting for x, y, z in terms of t

$$\overrightarrow{AB} = \begin{pmatrix} -1+2t \\ -t \\ 1+2t \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -5+2t \\ -t-3 \\ -1+2t \end{pmatrix}$$

Since $\overrightarrow{AB}$ is perpendicular to l and l is parallel to $(2,-1,2)$

$\overrightarrow{AB}\cdot(2,-1,2)=0$

Hence

$(-5+2t,-t-3,1+2t)\cdot(2,-1,2)=0$

$2(-5+2t)-(-t-3)+2(-1+2t)=0$

$-10+4t+t+3-2+4t=0$

$t=1$

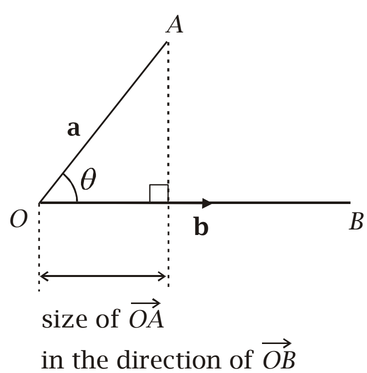
$$\overrightarrow{AB}=\begin{pmatrix}-5+2t\\-t-3\\-1+2t\end{pmatrix}=\begin{pmatrix}-5+2\\-1-3\\-1+2\end{pmatrix}=\begin{pmatrix}-3\\-4\\1\end{pmatrix}$$

A unit vector is

$$\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}=\frac{1}{\sqrt{26}}(-3,-4,1)$$

Projection of one vector onto another

The size of the vector $\mathbf{a}=\overrightarrow{OA}$ in the direction of another vector $\mathbf{b}=\overrightarrow{OB}$ is called the *orthogonal projection* of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$.



It is given by $\left|\overrightarrow{OP}\right|=\left|\mathbf{a}\cos\theta\right|=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$.

Example (7)

Find the orthogonal projection of $\mathbf{p}=(\mathbf{2},\mathbf{4},-\mathbf{5})$ onto $\mathbf{q}=(-1,3,-1)$.

$$\text{Projection of } \mathbf{p} \text{ onto } \mathbf{q} = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}|}$$

$$= \frac{(2,4,-5)\cdot(-1,3,-1)}{|(-1,3,-1)|}$$

$$= \frac{15}{\sqrt{11}}$$