Scalar Triple Product

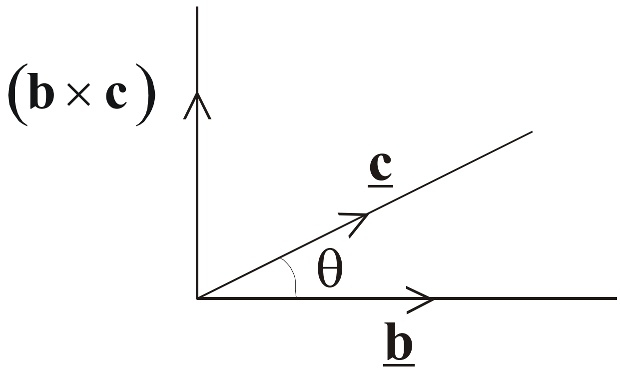
The scalar triple product is any expression of the form

$$\underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$$

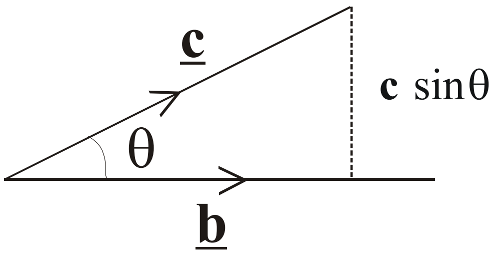
To understand what this expression "means" we need to revise firstly the meaning of the cross product.

The vector $\underline{\mathbf{b}} \times \underline{\mathbf{c}}$ is perpendicular to both the vectors $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$. If θ is the angle between these two vectors, $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$, then

$$|\underline{\mathbf{b}} \times \underline{\mathbf{c}}| = |\underline{\mathbf{b}}||\underline{\mathbf{c}}|\sin\theta$$



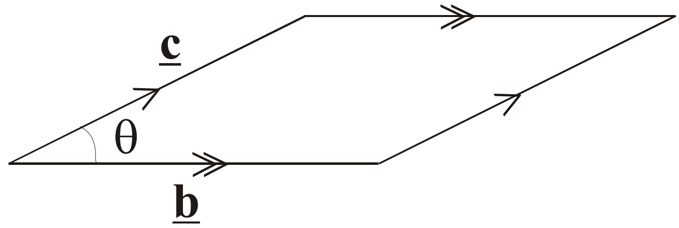
The expression $|\underline{\mathbf{c}}|\sin\theta$ denotes the "height" of the vector $\underline{\mathbf{c}}$.



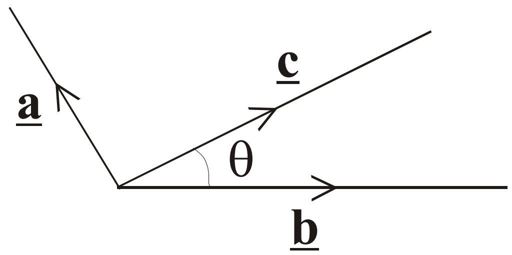
The area of a parallelogram is

$$\text{Area } = \text{base} \times \text{height}$$

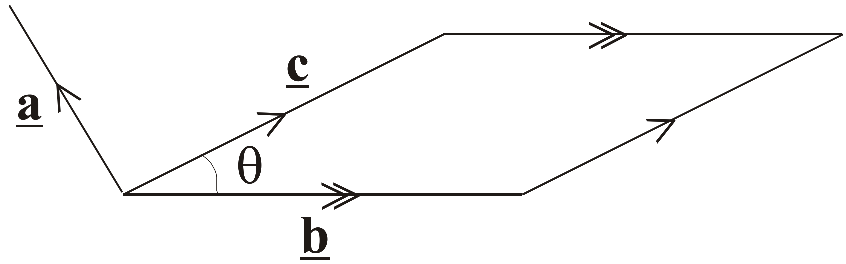
Here the base is $\underline{\mathbf{b}}$ and the height is $|\underline{\mathbf{c}}|\sin\theta$, so $|\underline{\mathbf{b}} \times \underline{\mathbf{c}}|$ is the area of the parallelogram with sides of length $|\underline{\mathbf{b}}|$ and $|\underline{\mathbf{c}}|$ making an angle θ.



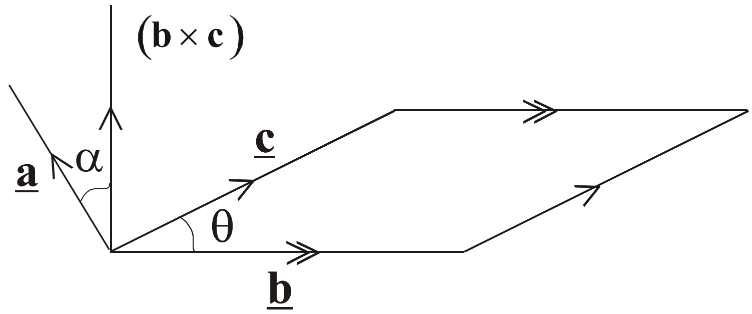
Now imagine that we have three vectors, $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$.



The magnitude of the vector $\underline{\mathbf{b}} \times \underline{\mathbf{c}}$, that is $\left|\underline{\mathbf{b}} \times \underline{\mathbf{c}}\right|$, is the area of the base of this figure, the parallelogram with $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$ as sides.



The vector $\underline{\mathbf{b}} \times \underline{\mathbf{c}}$ is perpendicular to both $\underline{\mathbf{b}}$ and $\underline{\mathbf{c}}$, and let α be the angle between this vector and the vector $\underline{\mathbf{a}}$.



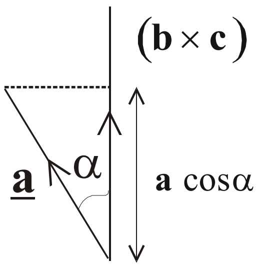
So, from the definition of the scalar (dot) product

$$\cos\alpha = \frac{\underline{\mathbf{a}} \cdot \left(\underline{\mathbf{b}} \times \underline{\mathbf{c}}\right)}{\left|\underline{\mathbf{a}}\right| \cdot \left|\underline{\mathbf{b}} \times \underline{\mathbf{c}}\right|}$$

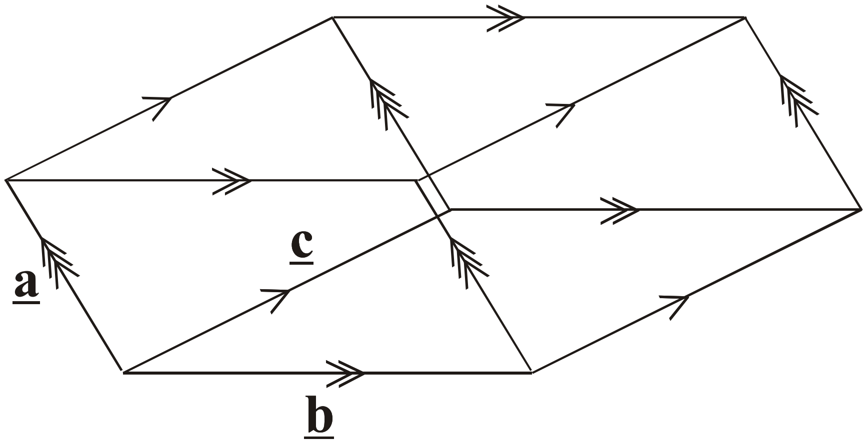
so

$\underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = |\underline{\mathbf{a}}| \cos\alpha \cdot |\underline{\mathbf{b}} \times \underline{\mathbf{c}}|$

The quantity $|\underline{\mathbf{a}}| \cos\alpha$ is the projection of the vector $\underline{\mathbf{a}}$ onto the vector $\underline{\mathbf{b}} \times \underline{\mathbf{c}}$



So the quantity $\underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$ is the volume of the quasi-rectangular object bounded by vectors **a**, **b** and **c** and other vectors parallel to these



$|\underline{\mathbf{b}} \times \underline{\mathbf{c}}|$ is the area of the base, and $|\underline{\mathbf{a}}| \cos\alpha$ is the height.

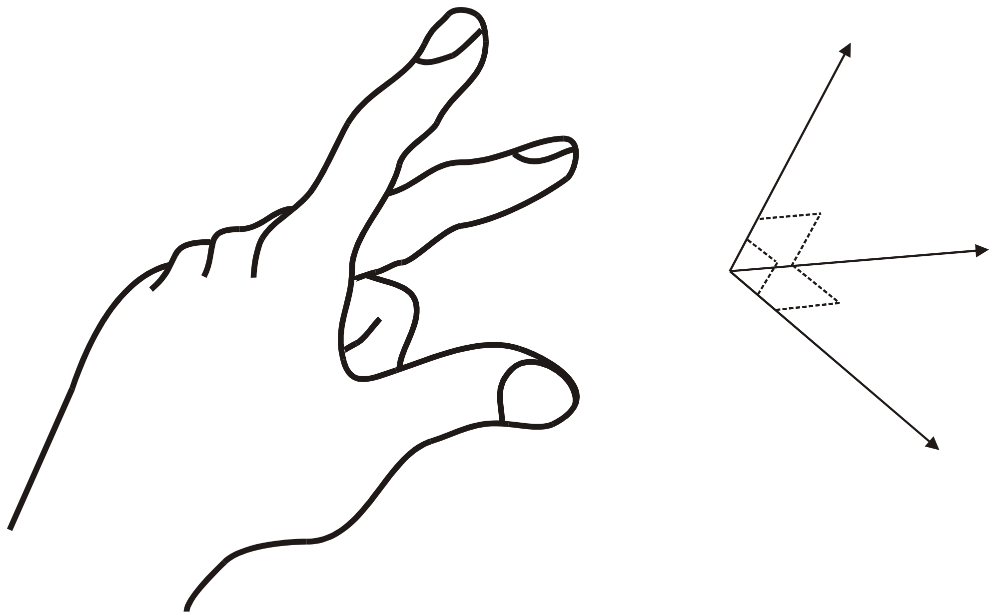
This figure is called a parallelipiped. It is a polyhedron with six faces, each of which is a parallelogram.

The triple scalar product as a "signed" volume

The value $\underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$ represents a "signed" volume. This means that it can be a positive or a negative quantity. Of course, we are aware that volumes do not strictly take a "sign". However, the sign that this quantity takes is a by-product of the way in which we calculate the volume using vectors. In practical examples it is often just the physical volume of the parallelepiped that is desired, so you simply "drop" the sign at the end of the process. However, the "abstract" meaning of the sign is that if the volume is positive then the vectors making up its edges form a "left-handed set", and if it is negative then they form a "right-handed set".

The meaning of "left-handed" and "right-handed" sets are defined as follows.

If you make your index, second finger and thumb point at right angles to each other, then you have a left-handed set of perpendicular vectors if you are using your left hand, and a right-handed set of perpendicular vectors if you are using your right hand!



The scalar-triple product gives an algebraic method of determining whether you have a left or right-hand set

right-hand set of vectors $\Leftrightarrow \underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) > 0$

left-hand set of vectors $\Leftrightarrow \underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) < 0$

Evaluating a triple scalar product

<u>Example</u>

Find $\underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$ where

$\underline{\mathbf{a}} = 2\underline{\mathbf{i}} - 3\underline{\mathbf{j}} + 4\underline{\mathbf{k}}$

$\underline{\mathbf{b}} = 4\underline{\mathbf{i}} + 5\underline{\mathbf{j}} - 3\underline{\mathbf{k}}$

$\underline{\mathbf{c}} = -3\underline{\mathbf{i}} + 4\underline{\mathbf{j}} - 5\underline{\mathbf{k}}$

Solution

$$\underline{\mathbf{a}}\cdot\left(\underline{\mathbf{b}}\times\underline{\mathbf{c}}\right)=\underline{\mathbf{a}}\cdot\begin{vmatrix}\underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}}\\ 4 & 5 & -3\\ -3 & 4 & -5\end{vmatrix}$$

$$=\underline{\mathbf{a}}\cdot\left(\underline{\mathbf{i}}\begin{vmatrix}5 & -3\\ 4 & -5\end{vmatrix}+\underline{\mathbf{j}}\begin{vmatrix}-3 & -5\\ 4 & -3\end{vmatrix}+\underline{\mathbf{k}}\begin{vmatrix}4 & 5\\ -3 & 4\end{vmatrix}\right)$$

$$=\underline{\mathbf{a}}\cdot\left(\underline{\mathbf{i}}\left(-25+12\right)+\underline{\mathbf{j}}\left(9+20\right)+\underline{\mathbf{k}}\left(16+15\right)\right)$$

$$=\underline{\mathbf{a}}\cdot\left(-13,29,31\right)$$

$$=\left(2,-3,4\right)\cdot\left(-13,29,31\right)=11$$

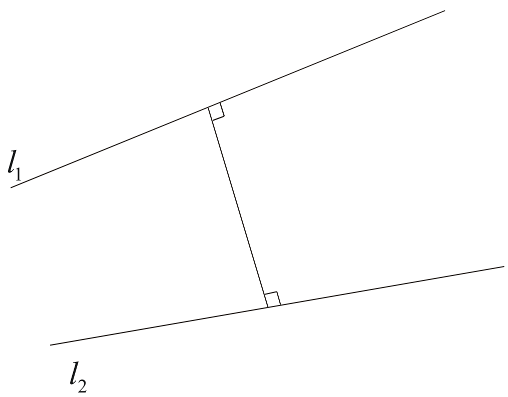
Note that the quantity $\underline{\mathbf{a}}\cdot\left(\underline{\mathbf{b}}\times\underline{\mathbf{c}}\right)$ is given by

$$\underline{\mathbf{a}}\cdot\left(\underline{\mathbf{b}}\times\underline{\mathbf{c}}\right)=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \qquad \text{where } \underline{\mathbf{a}}=\left(a_1,a_2,a_3\right) \text{ etc.}$$

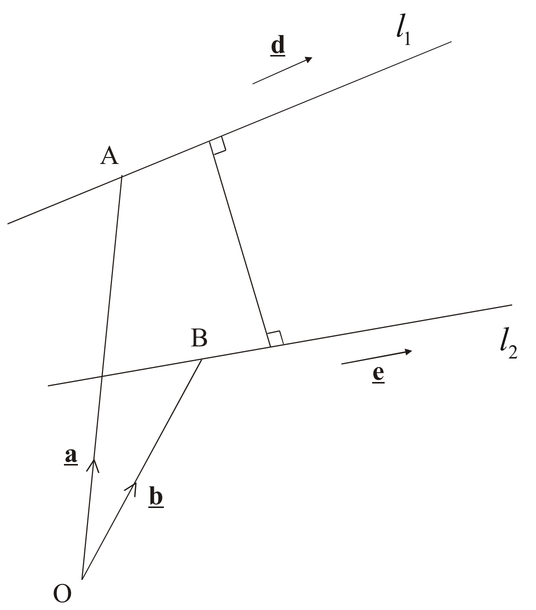
So finding a volume of a parallelepiped is equivalent to evaluating the determinant of a 3×3 matrix.

Shortest distance between two skew lines

The shortest distance between two lines is the length of the line that is perpendicular to both lines.



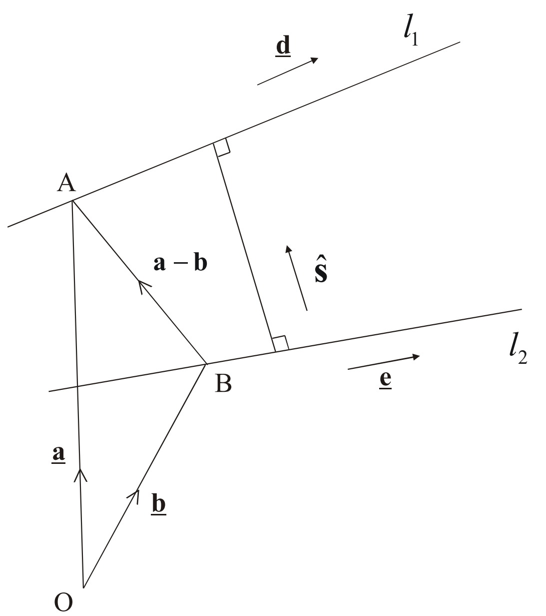
The vector equation of the line l_1 is $\underline{\mathbf{r}}=\underline{\mathbf{a}}+t\underline{\mathbf{d}}$, where $\underline{\mathbf{d}}$ is a vector parallel to the line, and $\underline{\mathbf{a}}$ is the position vector of a point A on the line. Likewise, the vector equation of the line l_2 is $\underline{\mathbf{r}}'=\underline{\mathbf{b}}+t'\underline{\mathbf{e}}$, where $\underline{\mathbf{e}}$ is a vector parallel to the line, and $\underline{\mathbf{b}}$ is the position vector of a point on the line.



The vector $\underline{\mathbf{s}} = \underline{\mathbf{d}} \times \underline{\mathbf{e}}$ is perpendicular to both $\underline{\mathbf{d}}$ and $\underline{\mathbf{e}}$. A unit vector, therefore, parallel to the shortest line between the two lines is give by

$$\underline{\hat{\mathbf{s}}} = \frac{\underline{\mathbf{s}}}{|\underline{\mathbf{s}}|} = \frac{\underline{\mathbf{d}} \times \underline{\mathbf{e}}}{|\underline{\mathbf{d}} \times \underline{\mathbf{e}}|}$$

Now the vector $\underline{\mathbf{a}} - \underline{\mathbf{b}}$ is the vector joining B to A.



If θ is the angle between $\underline{\hat{\mathbf{s}}}$ and $\underline{\mathbf{a}} - \underline{\mathbf{b}}$, then

$$|\underline{\mathbf{a}} - \underline{\mathbf{b}}|\cos\theta$$

is the projection of $\underline{\mathbf{a}} - \underline{\mathbf{b}}$ onto $\underline{\hat{\mathbf{s}}}$. Hence, this is the shortest distance between the two lines:

shortest distance = $|\underline{\mathbf{a}} - \underline{\mathbf{b}}|\cos\theta$.

Since,

$$\cos\theta = \frac{(\underline{\mathbf{a}} - \underline{\mathbf{b}}) \cdot \underline{\hat{\mathbf{s}}}}{|\underline{\mathbf{a}} - \underline{\mathbf{b}}||\underline{\hat{\mathbf{s}}}|}$$

Then,

$$|\underline{\mathbf{a}} - \underline{\mathbf{b}}|\cos\theta = (\underline{\mathbf{a}} - \underline{\mathbf{b}}) \cdot \underline{\hat{\mathbf{s}}}$$

That is,

$$\text{shortest distance} = (\underline{\mathbf{a}} - \underline{\mathbf{b}}) \cdot \frac{\underline{\mathbf{d}} \times \underline{\mathbf{e}}}{|\underline{\mathbf{d}} \times \underline{\mathbf{e}}|}$$

This may have a "sign" (positive or negative sign) depending on whether $\underline{\mathbf{s}} = \underline{\mathbf{d}} \times \underline{\mathbf{e}}$ forms a left or right-hand set. As we are only interested in the magnitude of the shortest distance, we take the modulus of the above expression

$$\text{shortest distance} = \left|(\underline{\mathbf{a}} - \underline{\mathbf{b}}) \cdot \frac{\underline{\mathbf{d}} \times \underline{\mathbf{e}}}{|\underline{\mathbf{d}} \times \underline{\mathbf{e}}|}\right|$$

This expression contains a triple scalar product.

Using this formula we can find the shortest distance between two skew lines. (Lines are skew when they form a non-zero angle between their directions, and when they do not intersect.

Example

Fin the shortest distance between the following two skew lines

$$\underline{\mathbf{r}} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \underline{\mathbf{r}}' = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t'\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

Solution

$$\underline{\mathbf{a}} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \qquad \underline{\mathbf{b}} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$$

Therefore,

$$\underline{\mathbf{a}} - \underline{\mathbf{b}} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix}$$

$$\underline{\mathbf{d}} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \qquad \underline{\mathbf{e}} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

Hence

$$\begin{aligned} \underline{\mathbf{d}} \times \underline{\mathbf{e}} &= \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ -1 & -1 & -1 \\ -1 & 1 & 2 \end{vmatrix} \\ &= \underline{\mathbf{i}} \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + \underline{\mathbf{j}} \begin{vmatrix} -1 & -1 \\ 2 & -1 \end{vmatrix} + \underline{\mathbf{k}} \begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} \\ &= \underline{\mathbf{i}}(-2+1) + \underline{\mathbf{j}}(1+2) + \underline{\mathbf{k}}(-1-1) \\ &= \begin{pmatrix} -1 & 3 & -2 \end{pmatrix} \end{aligned}$$

Hence

$$(\underline{\mathbf{a}} - \underline{\mathbf{b}}) \cdot (\underline{\mathbf{d}} \times \underline{\mathbf{e}}) = \begin{pmatrix} -1 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 3 & -2 \end{pmatrix} = 1 + 15 - 2 = 14$$

Intersecting and parallel lines

If two lines intersect then the shortest distance between them is zero. That means

$$\text{shortest distance} = (\underline{\mathbf{a}} - \underline{\mathbf{b}}) \cdot \left| \frac{\underline{\mathbf{d}} \times \underline{\mathbf{e}}}{|\underline{\mathbf{d}} \times \underline{\mathbf{e}}|} \right| = 0$$

Hence

$$(\underline{\mathbf{a}} - \underline{\mathbf{b}}) \cdot (\underline{\mathbf{d}} \times \underline{\mathbf{e}}) = 0$$

This provides a way of checking whether two lines meet.

Example

Show that the following two lines

l_1 $\quad \underline{\mathbf{r}} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

and

l_2 $\quad \underline{\mathbf{r}} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Solution

In terms of the formula $\left(\underline{\mathbf{a}} - \underline{\mathbf{b}}\right) \cdot \left|\frac{\underline{\mathbf{d}} \times \underline{\mathbf{e}}}{\left|\underline{\mathbf{d}} \times \underline{\mathbf{e}}\right|}\right|$ we have

$\underline{\mathbf{a}} = (2,\text{-}1,4)$
$\underline{\mathbf{b}} = (3,1,6)$
$\underline{\mathbf{d}} = (2,1,2)$
$\underline{\mathbf{e}} = (-1,1,0)$

$$\begin{aligned}
\underline{\mathbf{d}} \times \underline{\mathbf{e}} &= (2,1,2) \times (-1,1,0) \\
&= \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 2 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} \\
&= \underline{\mathbf{i}}\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} + \underline{\mathbf{j}}\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} + \underline{\mathbf{k}}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} \\
&= -2\underline{\mathbf{i}} - 2\underline{\mathbf{j}} + 3\underline{\mathbf{k}} \\
&= (-2,-2,3)
\end{aligned}$$

$$\underline{\mathbf{a}} - \underline{\mathbf{b}} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

Hence

$$(\underline{\mathbf{a}} - \underline{\mathbf{b}}) \cdot \left| \frac{\underline{\mathbf{d}} \times \underline{\mathbf{e}}}{|\underline{\mathbf{d}} \times \underline{\mathbf{e}}|} \right| = (-1.-2.-2) \cdot (-2,-2,3) = 2 + 4 - 6 = 0$$

This simplifies the process of finding the shortest distance between two skew lines. However, if the lines are parallel, then the formula breaks down, for we have a different reason why it evaluates to zero. This is because, when **d** and **e** are parallel $\underline{\mathbf{d}} \times \underline{\mathbf{e}} = 0$, so we cannot find a perpendicular vector by this method. So, strictly speaking, in the above example, we should comment firstly that the vectors

$$\underline{\mathbf{d}} = (2,1,2)$$
$$\underline{\mathbf{e}} = (-1,1,0)$$

are not parallel. (If they would be parallel, one would be the scalar multiple of the other.)

But when two lines are parallel, we have to revert to other methods to find the (constant) distance between them; as in the following example.

Example

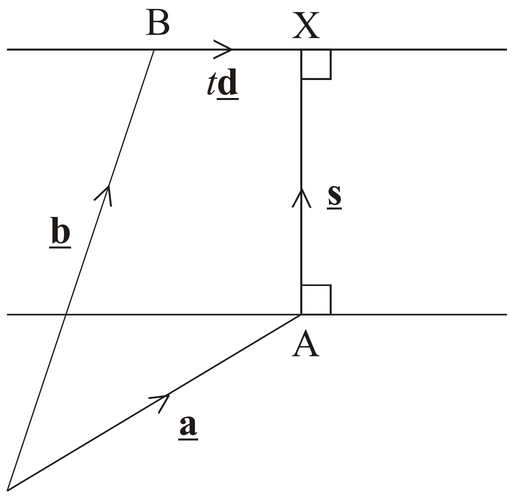
Find the shortest distance between the lines

l_1 $\quad \underline{\mathbf{r}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

and

l_2 $\quad \underline{\mathbf{r}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

Solution



Let $\underline{\mathbf{d}}$ be a vector parallel to the lines; for instance, here

$$\underline{\mathbf{d}} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$\underline{\mathbf{a}}$ is the position vector of a point A on the line l_1 and $\underline{\mathbf{b}}$ is the position vector of a point B on the second line l_2.

$\underline{\mathbf{s}}$ is a vector perpendicular to both lines.

Then

$$\underline{\mathbf{b}} + \underline{\mathbf{s}} = \underline{\mathbf{a}} + t\underline{\mathbf{d}}$$

and, because $\underline{\mathbf{s}}$ and $\underline{\mathbf{d}}$ are perpendicular

$$\underline{\mathbf{s}} \cdot \underline{\mathbf{d}} = 0$$

We solve these two equations simultaneously to obtain an expression for the vector $\underline{\mathbf{s}}$. The length of this vector will be the shortest distance. Thus, let

$$\underline{\mathbf{s}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \underline{\mathbf{a}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \underline{\mathbf{b}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \underline{\mathbf{d}} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

then

$\underline{\mathbf{b}} + \underline{\mathbf{s}} = \underline{\mathbf{a}} + t\underline{\mathbf{d}}$
implies

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

which, on uncoupling, gives

$x = 1 + 2t$
$y = 1 + 2t$
$1 + z = 1 + 2t$
The last expression simplifies to
$z = 2t$
Hence,
$x = 1 + z$
$y = 1 + z$

$\underline{\mathbf{s}} \cdot \underline{\mathbf{d}} = 0$
implies

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 0$$

which, on uncoupling, gives

$x + y + z = 0$
substituting $x = 1 + z$, and $y = 1 + z$ gives
$z + (1 + z) + (1 + z) = 0$
$3z = -2$
$z = -2/3$
Then since $z = 2t$,
$t = -1/3$
Hence,
$x = 1 + 2t = 1/3$
$y = 1 + 2t = 1/3$
Hence

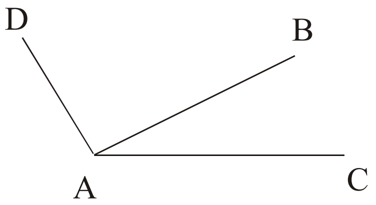
$$\underline{\mathbf{s}} = \begin{pmatrix} \tfrac{1}{3} \\ \tfrac{1}{3} \\ -\tfrac{2}{3} \end{pmatrix}$$

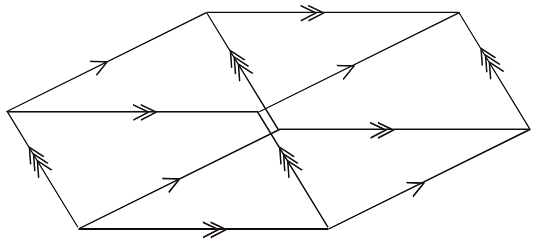
Hence, the shortest distance between the two parallel lines is

$$|\underline{\mathbf{s}}| = \left|\left(\tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right)\right| = \sqrt{\tfrac{1}{9} + \tfrac{1}{9} + \tfrac{4}{9}} = \sqrt{\tfrac{6}{9}} = \sqrt{\tfrac{2}{3}}$$

Co-planar points

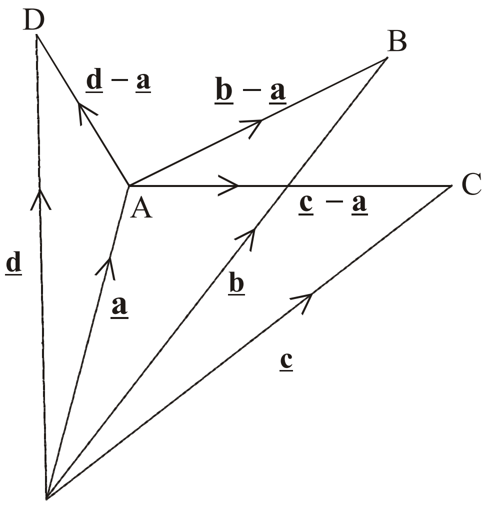
If four points are not planar, they define a parallelepiped with non-zero volume.





So, if the volume of the parallelepiped defined by four points is zero, then the parellelepiped has "collapsed" and the points must be coplanar.

If the position vectors of the points are **a**, **b**, **c** and **d** respectively, then the requirement that the volume of the parallelepiped be zero gives



$$(\underline{\mathbf{d}} - \underline{\mathbf{a}}) \cdot \left\{ \frac{(\underline{\mathbf{b}} - \underline{\mathbf{a}}) \times (\underline{\mathbf{c}} - \underline{\mathbf{a}})}{\left|(\underline{\mathbf{b}} - \underline{\mathbf{a}}) \times (\underline{\mathbf{c}} - \underline{\mathbf{a}})\right|} \right\} = 0$$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