# Solution of second order homogeneous constant coefficient differential equations

## Prerequisites

You should already be familiar with concept of a differential equation and the solution of first order differential equations by separation of variables. It is also an advantage to have studied complex numbers and be familiar with Euler's formula, which requires familiarity with standard Taylor series expansions of $e^x$, $\sin x$ and $\cos x$. It is helpful to understand what is meant by a linear combination of functions, the term "linear independence" and also to have studied sinusoidal functions.

**Example (1)**

Use the technique of separation of variables to solve $\frac{x}{t}\frac{dx}{dt} = \frac{1}{1+t^2}$. Find the particular solution if $x = 1$ when $t = 0$.

Solution

$$\frac{x}{t}\frac{dx}{dt} = \frac{1}{1+t^2}$$

$$\int x\,dx = \int \frac{t}{1+t^2}\,dt$$

$$\frac{x^2}{2} = \frac{1}{2}\ln\left(1+t^2\right) + c$$

$$x^2 = \ln\left(1+t^2\right) + c_1$$

Substituting $t = 0, x = 1$

$$1 = \ln(1) + c_1 \qquad \Rightarrow \qquad c_1 = 1$$

Hence the particular solution is

$$x^2 = \ln\left(1+t^2\right) + 1$$

**Example (2)** (optional)

Let $am^2 + bm + c = 0$ where $\Delta = b^2 - 4ac < 0$. Prove that the roots of this equation are complex conjugates.

Solution

$am^2 + bm + c = 0$

From the quadratic formula

$$m_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

In this $\sqrt{b^2 - 4ac} = \delta i$ is an imaginary number $(\delta \text{ is real})$ since $\Delta = b^2 - 4ac < 0$. So the roots have the form

$m_{1,2} = \alpha \pm \beta i$

where $\alpha = \frac{-b}{2a}$ and, if $\delta i = \sqrt{b^2 - 4ac}$, then $\beta = \frac{\delta}{2a}$

**Example (3)** (optional)

The standard Taylor series for $e^x$, $\sin x$, and $\cos x$ are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$$

Use these to prove Euler's formula

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

Solution

Let $z = x + iy$, then

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x \left\{ 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + ... \right\} \\ &= e^x \left\{ 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} - ... \right\} \quad \text{since } i^2 = -1 \\ &= e^x \left\{ \left( 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - ... \right) + i \left( y - \frac{y^3}{3!} + \frac{y^5}{5!} - ... \right) \right\} \\ &= e^x (\cos y + i \sin y) \end{aligned}$$

# Classification of differential equations

In a *differential equation* we have an equation in the dependent variable $x$ given in terms of various rates of change of the independent variable $t$. A differential equation is an equation involving such symbols as

$$\frac{dx}{dt} \qquad \frac{d^2x}{dt^2} \qquad \frac{d^3x}{dt^3} \qquad\qquad f'(t) \qquad f''(t) \qquad f'''(t)$$

and so forth. The general aim is to find *exact solutions* to such differential equations. The exact solution is a family of functions of the form $x = f(t) + c$, where $c$ is a constant that satisfies the differential equation. Exact solutions to differential equations cannot always be found. When they cannot be found, the solution may be approximated by a numerical method. Differential equations differ from each other in various systematic ways, and their exact solutions, when they can be found, differ accordingly. A differential equation is called *first-order* if the highest derivative is a first derivative – that is, of the form $\frac{dx}{dt}$. It is called *second-order* if the highest derivative is a second derivative, of the form $\frac{d^2x}{dt^2}$. The general form of a second-order differential equation is

$$p(t)\frac{d^2x}{dt^2} + q(t)\frac{dx}{dt} + r(t)x = f(t)$$

where $p(t), q(t), r(t)$ and $f(t)$ are functions of the independent variable $t$. When $f(t) \equiv 0$ the equation is *homogeneous*. Otherwise, the equation is said to be *nonhomogeneous* (or inhomogeneous). If $p(t), q(t)$ and $r(t)$ are constant functions then we obtain a second-order, homogeneous, *constant coefficient* differential equation.

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

**Example (4)**

Classify each of the following differential equations as (1) first-order or second-order; (2) homogenous or nonhomogeneous; (3) constant coefficient or non-constant coefficient.

$(a)$ $\frac{dx}{dt} = 5 + t^2$ $\qquad$ $(b)$ $3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 3x + \sin x = 0$

$(c)$ $\sin x \frac{dx}{dt} = x^2 + e^t$ $\qquad$ $(d)$ $\frac{d^2y}{dx^2} = -kx$

$(e)$ $\frac{d^2s}{dt^2} - 4\cos t\frac{ds}{dt} + 6 = 0$ $\qquad$ $(f)$ $e^x\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 3x = \sin(x+3)$

Solution

$(a)$ $$\frac{dx}{dt}=5+t^2$$

First order, constant coefficient, nonhomogeneous

$(b)$ $$3\frac{d^2x}{dt^2}+5\frac{dx}{dt}-3x+\sin x=0$$

Rearrange to

$$3\frac{d^2x}{dt^2}+5\frac{dx}{dt}-3x=-\sin x$$

Second order, constant coefficient, nonhomogeneous

$(c)$ $$\sin x\frac{dx}{dt}=x^2+e^t$$

Rearrange to

$$\sin x\frac{dx}{dt}-x^2=e^t$$

First order, non constant coefficient, nonhomogeneous

$(d)$ $$\frac{d^2y}{dx^2}=-kx$$

Rearrange to

$$\frac{d^2y}{dx^2}+kx=0$$

Second order, constant coefficient, homogeneous

$(e)$ $$\frac{d^2s}{dt^2}-4\cos t\frac{ds}{dt}+6=0$$

Second order, non constant coefficient, homogeneous

$(f)$ $$e^x\frac{d^2x}{dt^2}+5\frac{dx}{dt}-3x=\sin(x+3)$$

Second order, not constant coefficient, nonhomogeneous

In this chapter we shall be concerned with finding solutions to second order, constant coefficient homogeneous differential equations. That is, equations of the type

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=0$$

where *a, b* and *c* are constants.

# Solution to second order, constant coefficient, homogeneous differential equations

To solve $a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=0$ where *a*, *b* and *c* are constants. There is a complete solution to this type of equation. It is given by the following procedure.

**(1) Form the auxiliary equation**

Form the auxiliary equation by replacing the term in $\frac{d^2x}{dt^2}$ by $m^2$, the term in $\frac{dx}{dt}$ by *m* and the term in *x* by 1 to obtain an equation of the form

$am^2+bm+c=0$

**Example (5)**

Find the auxiliary equation to the differential equation $\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=0$.

Solution

$m^2+5m+6=0$

**(2) Solve the auxiliary equation**

The auxiliary equation is a quadratic equation in *m*. Solve this equation either directly or by using the quadratic formula to obtain its two roots $m_1$ and $m_2$.

**Example (5) continued**

Find the roots of the equation $m^2+5m+6=0$.

Solution

$m^2+5m+6=0$

$(m+3)(m+2)=0$

The roots are

$m_1=-3$ and $m_2=-2$

**(3) Classify the auxiliary equation and identify the form of the solution**

The roots of the auxiliary equation $am^2+bm+c=0$ may be classified into one of three cases, depending on whether its discriminant $\Delta=b^2-4ac$ is positive, zero or negative.

(A) $\Delta > 0$

The roots $m_1$ and $m_2$ are real and distinct. Then the solution to the original differential equation takes the form

$$x(t) = Ae^{m_1 t} + Be^{m_2 t}$$

where $A$ and $B$ are constants whose particular values may be determined by given additional boundary conditions.

(B) $\Delta = 0$

There is only one solution to the auxiliary equation. So we have $m_1 = m_2$ and the root is real and repeated. Then the solution to the original differential equation takes the form

$$x(t) = (A + Bt)e^{m_1 t}$$

where $A$ and $B$ are constants whose particular values may be determined by given additional boundary conditions.

(C) $\Delta < 0$

The roots $m_1$ and $m_2$ are conjugate complex numbers where

$$m_1 = \alpha + i\beta$$
$$m_2 = \alpha - i\beta$$

Then the solution to the original differential equation takes the form

$$x(t) = e^{\alpha t}(A\sin\beta t + B\cos\beta t)$$

where $A$ and $B$ are constants whose particular values may be determined by given additional boundary conditions.

**Example (5) continued**

Find the general solution to the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

Solution

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$
$$m^2 + 5m + 6 = 0$$
$$(m+3)(m+2) = 0$$
$$m_1 = -3 \text{ and } m_2 = -2$$

We have two real distinct solutions. The general form of the solution is $x(t) = Ae^{m_1 t} + Be^{m_2 t}$ and on substitution for $m_1$ and $m_2$ into this

$$x = Ae^{-3t} + Be^{-2t}$$

**(4) If initial conditions are given find the particular solution**

To find the values of the coefficients $A$ and $B$ in the general solution we require initial conditions (also called boundary conditions). If these are given, then differentiate the general solution and substitute accordingly to obtain two simultaneous equations in $A$ and $B$ that may be solved.

**Example (5) continued**

Find the particular solution to the differential equation

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=0$$

given $x=1$ and $\frac{dx}{dt}=2$ when $t=0$

Solution

We have shown that the general solution is

$$x=Ae^{-3t}+Be^{-2t} \qquad (1)$$

Differentiating this gives

$$\frac{dx}{dt}=-3Ae^{-3t}-2Be^{-2t} \qquad (2)$$

Substituting t = 0, $x=1$ and $t=0$, $\frac{dx}{dt}=2$ into (1) and (2) respectively we obtain

$$1=A+B$$
$$2=-3A-2B$$

These two simultaneous equations may be solved to give

$A=-4 \quad B=5$

Hence, the particular solution is

$$x=-4e^{-3t}+5e^{-2t}$$

**Example (6)**

Solve $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$

Solution

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$$

$$m^2-6m+9=0$$

$$(m-3)^2=0$$

$$m_1=m_2=3$$

We have repeated real roots. The solution is

$y = (A + Bx)e^{3x}$

To find the constants $A$ and $B$ for a particular solution we would require initial conditions.

**Example (7)**

($a$) Solve $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13t = 0$

($b$) Find the particular solution to this equation given $x = -3$ and $\frac{dx}{dt} = 18$ when $t = 0$

Solution

($a$) $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13t = 0$

$m^2 + 4m + 13 = 0$

$m = \frac{-4 \pm \sqrt{16 - 52}}{2}$

$= \frac{-4 \pm 6i}{2}$

$= -2 \pm 3i$

We have conjugate, complex roots. The solution is found by substituting into

$x(t) = e^{\alpha t}(A \sin \beta t + B \cos \beta t)$

$\alpha = -2 \quad \beta = 3$

$x(t) = e^{-2t}(A \sin 3t + B \cos 3t)$

Where $A$ and $B$ are constants.

($b$) $x(t) = e^{-2t}(A \sin 3t + B \cos 3t)$

$t = 0, x = -3 \Rightarrow B = -3$

$\frac{dx}{dt} = -2e^{-2t}(A \sin 3t + B \cos 3t) + e^{-2t}(3A \cos 3t - 3B \sin 3t)$

Substituting $t = 0, \frac{dx}{dt} = 18$

$18 = -2B + 3A$

$18 = 6 + 3A$

$A = 4$

Hence, the particular solution is

$x_p(t) = e^{-2t}(4 \sin 3t - 3 \cos 3t)$

# Sinusoidal functions

To recap: given the differential equation $a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=0$ $(a, b, c \text{ constant})$ we form the auxiliary equation $am^2+bm+c=0$. When the discriminant of this equation $\Delta=b^2-4ac<0$, the roots $m_1$ and $m_2$ are conjugate complex numbers where $m_1=\alpha+i\beta$ and $m_2=\alpha-i\beta$ and the solution to the original differential equation takes the form $x(t)=e^{\alpha t}(A\sin\beta t+B\cos\beta t)$ where $A$ and $B$ are constants. You should be aware from your study of sinusoidal functions that a function of the form $A\sin\theta t+B\cos\theta t$ may be combined into a **single** sinusoidal function of the form

$A\cos\theta+B\sin\theta\equiv R\sin(\theta+\phi)$ $\quad$ $A\cos\theta+B\sin\theta\equiv R\cos(\theta+\phi')$

where $R$ is the amplitude and $\phi$ or $\phi'$ is the phase shift. Let us consolidate this.

**Example (7) continued**

Write $y(t)=(4\sin3t-3\cos3t)$ in the form

($a$) $R\sin(\theta+\phi)$ $\quad$ ($b$) $R\cos(\theta+\phi')$

Solution

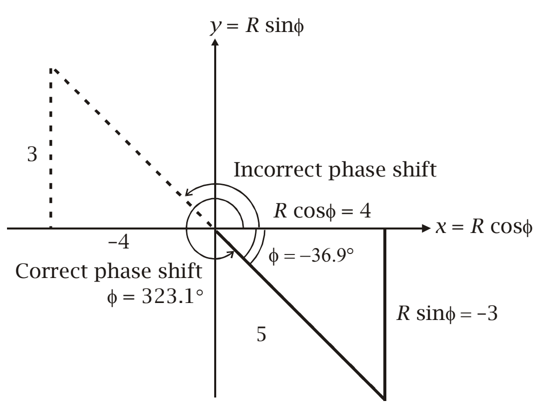
($a$) Let $y(t)=(4\sin3t-3\cos3t)\equiv R\sin(3t+\phi)$. Hence, expanding $R\sin(3t+\phi)$ by the usual compound angle formula

$y(t)=(4\sin3t-3\cos3t)\equiv R\sin(3t)\cos\phi+R\cos(3t)\sin\phi$.

Whence, by equating coefficients: $\quad R\cos\phi=4$ and $R\sin\phi=-3$

$R=\sqrt{(4)^2+(-3)^2}=5$ $\quad$ $\phi=\tan^{-1}\left(\frac{-3}{4}\right)=-36.9°$ ( or $323.1°$) $(0.1°)$

In general there are two choices of the angle $\phi$ corresponding to the value $\phi=\tan^{-1}\left(\frac{-3}{4}\right)$. Only one of these is correct.



Hence $y(t) = (4\sin 3t - 3\cos 3t) \equiv 5\sin(3t - 36.9°)$

(*a*) Let $y(t) = (4\sin 3t - 3\cos 3t) \equiv R\cos(3t + \phi')$. Hence, expanding $R\cos(3t + \phi')$ by the usual compound angle formula

$$y(t) = (4\sin 3t - 3\cos 3t) \equiv R\cos(3t)\cos\phi - R\sin(3t)\sin\phi .$$

Whence, by equating coefficients: $R\cos\phi = -3$ and $R\sin\phi = -4$

$$R = \sqrt{(-3)^2 + (-4)^2} = 5 \qquad \phi = \tan^{-1}\left(\frac{-4}{-3}\right) = -126.9° \quad (0.1°)$$

Hence $y(t) = (4\sin 3t - 3\cos 3t) \equiv 5\cos(3t - 126.9°)$

We can sketch the graph of $y(t)$ given in example (7). Note that the two forms the function takes

$y(t) = 5\sin(3t - 36.9°)$

$y(t) = 5\cos(3t - 126.9°)$

give the same graph. They are identical. This is because a sine function arises from the horizontal translation of the cosine function by 90°

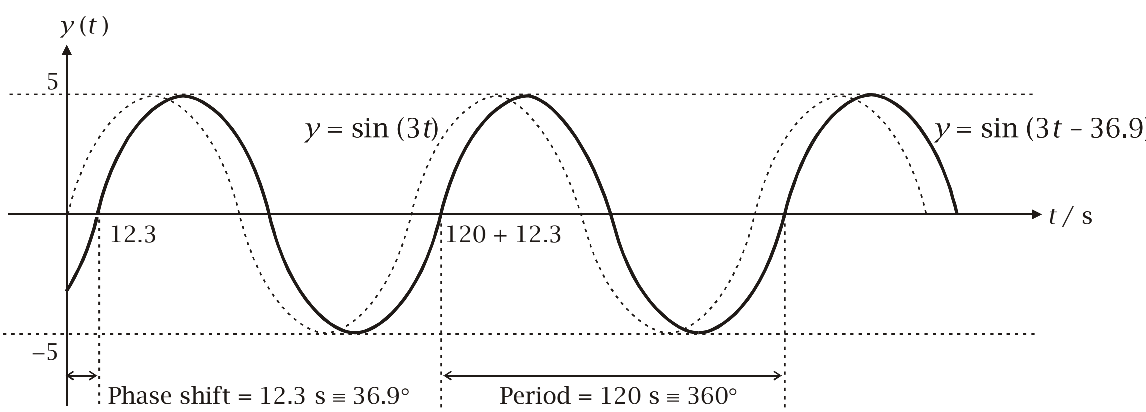
$\sin\theta \equiv \cos(\theta - 90°)$ [horizontal translation of $\cos\theta$ by 90°]

$\cos\theta \equiv \sin(\theta + 90°)$ [horizontal translation of $\sin\theta$ by $-90°$]

whence

$y(t) = 5\sin(3t - 36.9) = 5\cos(3t - 36.9 - 90) = 5\cos(3t - 126.9)$

Let us suppose that the independent variable $t$ in $y(t) = 5\sin(3t - 36.9)$ represents time in seconds and angles are measured in degrees. Then the period of the function $5\sin(3t)$ is 120 seconds. This is because it takes 120 seconds to sweep out the equivalent angle of 360°. When $t = 12.3$ s then $y(t) = 5\sin(0) = 0$, so the phase shift is $t = 12.3\text{ s} \equiv 36.9°$. The maximum value of $y(t) = 5\sin(3t - 36.9)$, called the *amplitude*, is 5.



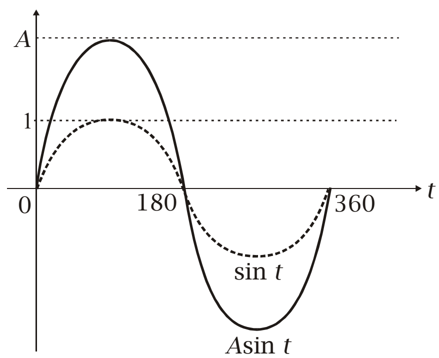
**Transformations of sine curves**

You should be aware that all sine curves have essentially the same shape, and hence may be derived from the function $x(t) = \sin t$ by a sequence of transformations.

**(1) Scaling in the vertical axis**

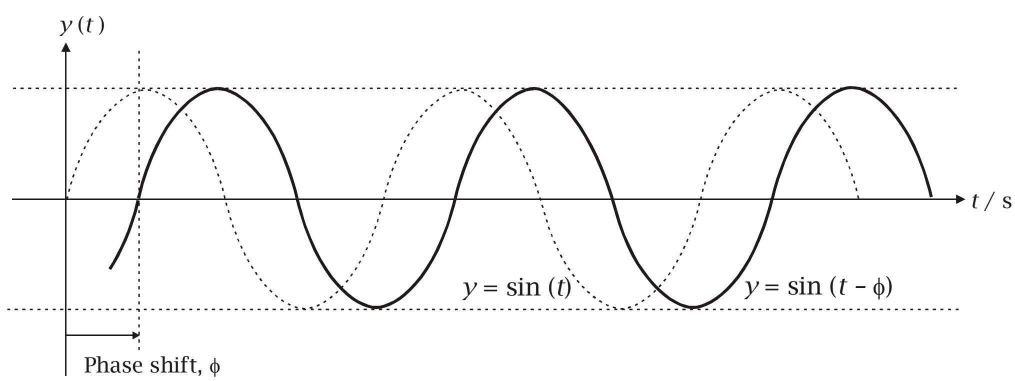
$x(t) = A\sin t$

A sine curve with amplitude $A$




(2) **A phase shift in a horizontal translation**

$x(t) = \sin(t - \phi)$



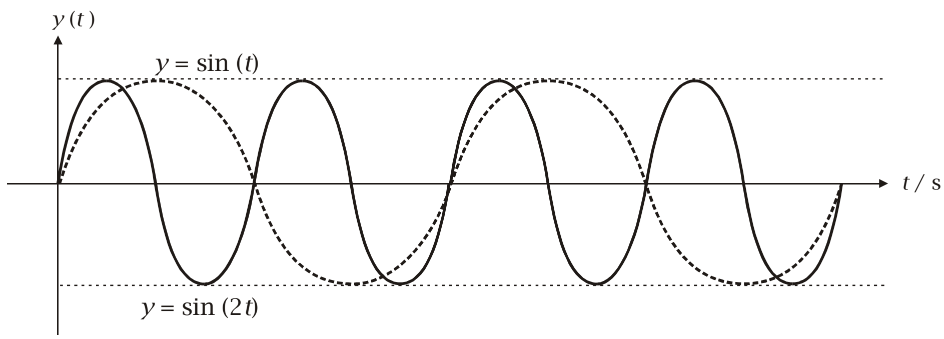

The oscillation represented by $x(t) = \sin(t - \phi)$ starts $\phi$ seconds *after* the oscillation $x(t) = \sin t$, and $\phi$ is called the phase shift of the oscillation. Note that the shift is in a positive direction (direction of increasing *t*) when the phase shift term is subtracted from the argument $(t)$ of the function. This is the same as with all translations of functions in a horizontal direction.

(3) Scaling in the horizontal axis (time)

If the independent variable $t$ represents time, this creates the sine curve of an oscillation with a different speed

$x(t) = \sin \omega t$

is a sinusoidal function with period $\frac{360}{\omega}$ seconds (where angles are measured in degrees). It represents an oscillation that is $\omega$ faster than $x(t) = \sin t$. The term $\omega$ is called the *angular velocity* of the oscillation. It is the angle swept out in 1 second. Angles may be measured in degrees $(°)$ or radians $(^{c}$ or rad$)$.




**Example (7) continued**

How may $y(t) = (4\sin 3t - 3\cos 3t)$ be obtained from $x(t) = \sin t$ by three transformations? Given $t$ represents time in seconds, and angles are measured in degrees, what are the amplitude, phase shift and angular velocity of $y(t)$?

Solution

We have shown already $y(t) = (4\sin 3t - 3\cos 3t) \equiv 5\sin(3t - 36.9)$

Thus, $y(t)$ is obtained from $x(t) = \sin t$ by

(1) A horizontal scaling of 3, equivalent to a threefold increase in the speed of $x(t) = \sin t$, followed by

(2) A phase shift of +36.9° from $x(t) = \sin t$, followed by

(3) A vertical sacling by 5; equivalent to a fivefold increase of the amplitude of $x(t) = \sin t$.

The amplitude, phase shift and angular velocity of $y(t)$ are

$A = 5$ m $\qquad \phi = 36.9° \equiv 12.3$ s $\qquad \omega = 3$ °s$^{-1}$

**Summary**

In conclusion a sinusoidal function of the form

$y(t) = A\sin\theta t + B\cos\theta t \quad A, B$ constants

may be written as

$y(t) = R\sin(\omega t + \phi)$

where $R$ is the amplitude, $\omega$ is the angular velocity, and $\phi$ is the phase shift.

If $t$ is time (seconds) and angles are measured in degrees, then the period is $T = \frac{360}{\omega}$ seconds.

If $t$ is time (seconds) and angles are measured in radians, then the period is $T = \frac{2\pi}{\omega}$ seconds.

# The damping factor

We have seen that the solution to a second order homogeneous constant coefficient differential equation may take the form

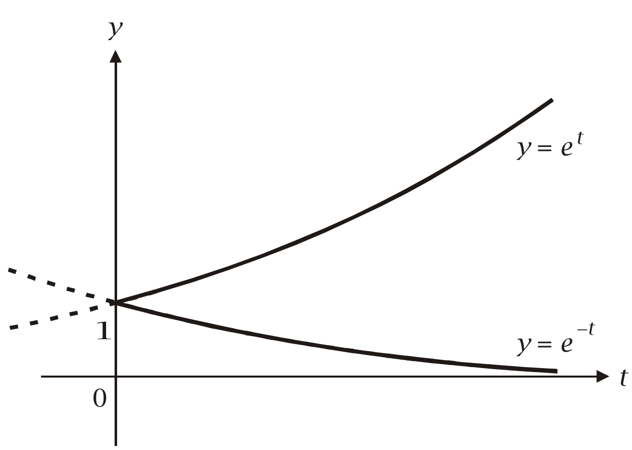
$x(t) = e^{\alpha t}(A\sin\beta t + B\cos\beta t)$

when the roots of the auxiliary equation are complex. In the preceding section we were able to interpret the part in this function given by

$A\sin\beta t + B\cos\beta t \equiv R\sin(\beta t + \phi)$

as a sinusoidal function with amplitude $R$, angular velocity $\beta$ and phase shift $\phi$. Now it remains to interpret the meaning of the $e^{\alpha t}$ part in $x(t) = e^{\alpha t}(A\sin\beta t + B\cos\beta t)$.

The function $f(t) = e^{\alpha t}$ is an exponential function. If $\alpha > 1$ this is upward sloping; if $\alpha < 1$ it is downward sloping.



In $x(t) = e^{\alpha t}(A\sin\beta t + B\cos\beta t)$ the two functions

$e^{\alpha t}$ $\qquad A\sin\beta t + B\cos\beta t$

are multiplied together. Therefore, the oscillations represented by

$A\sin\beta t + B\cos\beta t \equiv R\sin(\beta t + \phi)$

are subject to a vertical scale factor represented by $e^{\alpha t}$. If $\alpha > 1$ the oscillations are increasing. If $\alpha < 1$ the oscillations are decreasing, and the system is subject to *damping*. The term $\alpha$ is called the *damping factor*.

**Example (7) continued**

In example (7) we found the particular solution to a differential equation to be

$x_p(t) = e^{-2t}(4\sin 3t - 3\cos 3t)$

Take $t$ to represent time and assume angles are measured in degrees.

(*a*) Find the damping factor of $x_p(t)$.

(*b*) Describe the physical response of the system represented by $x_p(t)$.

(*c*) Sketch the graph of $x_p(t)$.

Solution

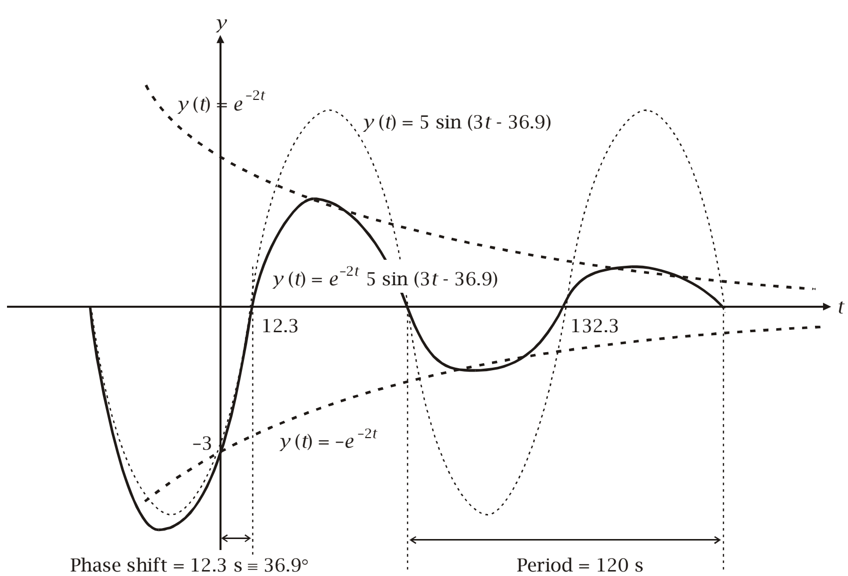
(*a*) $x_p(t) = e^{-2t}(4\sin 3t - 3\cos 3t)$

The damping factor is $\alpha = -2$

(*b*) The system is subject to damping; the oscillations are decreasing.

(*c*) We have

$y(t) = e^{-2t}(4\sin 3t - 3\cos 3t) \equiv e^{-2t}5\sin(3t - 36.9)$



Regarding this graph, note that because of the damping factor the maxima and minima of $y(t) = e^{-2t}(4\sin 3t - 3\cos 3t) \equiv e^{-2t} 5\sin(3t - 36.9)$ occur before the corresponding peaks of the undamped function $y(t) = 5\sin(3t - 36.9)$. Also the graph of the damping function $e^{-2t}$ is not in fact tangent to $y(t)$.

# Towards a proof of the solution to a homogeneous, second-order differential equation

This section is optional. A rigorous proof of the results cited in this chapter is beyond its scope. However, we can go a long way to justifying and explaining why the solution is the solution. To do so, suppose we wish to solve

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + ct = 0$$

where $x(t)$ is a function of *t*. This equation is a linear, second order differential equation with constant coefficients. It is linear because the various derivatives and functions in it are added. (We do not have in it expressions such as $\frac{dx}{dt} \times t$ which is a product of a function or derivative of $x(t)$ and *t*.) Linearity is an important assumption of the background theory. To solve this equation, we assume one particular solution and use it to find the general solution. We suppose the particular solution is $x(t) = e^{mt}$. Then by differentiating twice we obtain

$$\frac{dx}{dt} = me^{mt}$$

$$\frac{d^2x}{dt^2} = m^2e^{mt}$$

Substitution of these formulae into the original equation gives

$$am^2e^{mt} + bme^{mt} + ce^{mt} = 0$$

On cancelling through by $e^{mt}$ we obtain the auxiliary equation

$$\boxed{am^2 + bm + c = 0} \qquad \text{AUXILIARY EQUATION}$$

Which has roots $m_1$ and $m_2$ So $e^{m_1 t}$ and $e^{m_2 t}$ will be solutions to the original differential equation. To see how these two solutions combine we need the following theorem.

**Theorem**

If the functions $f_1$ and $f_2$ are solutions to a homogeneous linear differential equation then any linear combination of $f_1$ and $f_2$ is also a solution. (By a linear combination of two functions $f_1$ and $f_2$ we mean a function of the form $a_1f_1+a_2f_2$ where $a_1$ and $a_2$ are constants.)

So this theorem tells us that the general solution of the differential equation is

$\boxed{x(t) = Ae^{m_1t} + Be^{m_2t}}$ GENERAL SOLUTION

where $A$ and $B$ are constants. The solution is not a particular function but a family of functions. The family is spanned by two linearly independent functions, so it is called a space – in fact, it is a *vector space*. There is an ambiguity in the general solution, because the roots of the auxiliary equation can take three forms according to whether the discriminant $\Delta = b^2 - 4ac$ of the auxiliary equation is positive, zero or negative.

($A$) $\Delta > 0$

$\boxed{x(t) = Ae^{m_1t} + Be^{m_2t}}$ General solution when $m_1 \neq m_2$ are REAL and DISTINCT

$A$ and $B$ constants.

($B$) $\Delta = 0$

$\boxed{x(t) = (A + Bt)e^{m_1t}}$ General solution when $m_1$ is REAL and REPEATED

$A$ and $B$ constants.

($C$) $\Delta < 0$

The roots $m_1$ and $m_2$ are conjugate complex numbers where

$m_1 = \alpha + \beta i \qquad m_2 = \alpha - \beta i$

$$\begin{aligned}
x(t) &= Ae^{(\alpha+\beta i)t} + Be^{(\alpha-\beta i)t} \\
&= Ae^{\alpha t}e^{\beta it} + Be^{\alpha t}e^{-\beta it} \\
&= e^{\alpha t}\left\{Ae^{\beta it} + Be^{-\beta it}\right\} \\
&= e^{\alpha t}\left\{A\cos\beta t + Ai\sin\beta t + B\cos(-\beta t) + iB\sin(-\beta t)\right\} \quad [\text{by Euler's Theorem}] \\
&= e^{\alpha t}\left\{(A+B)\cos\beta t + i(A-B)\sin\beta t\right\} \\
&= e^{\alpha t}\left\{A'\cos\beta t + B'\sin\beta t\right\}
\end{aligned}$$

So the general solution in this case is

$\boxed{x(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t)}$ General solution for $m_1$, $m_2$ COMPLEX

$A$ and $B$ constants.

This is not a rigorous proof of these results because we have to prove the theorem cited above and also a more general theorem that the number of linearly independent solutions there are to a homogeneous, linear differential equation, is equal to its degree. We also have to justify the solution where the root of the auxiliary is real and repeated.