Solution of second order inhomogeneous constant coefficient differential equations

Prerequisites

You should already be familiar with the solution of homogeneous, constant coefficient second order differential equations.

Second order homogeneous constant coefficient differential equations

Given $a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=0$ where *a, b* and *c* are constants, the auxiliary equation is

$am^2+bm+c=0$

with roots m_1 and m_2 and discriminant $\Delta=b^2-4ac$.

If $\Delta>0$ the roots m_1 and m_2 are real and distinct. The solution to the original differential equation takes the form $x(t)=Ae^{m_1t}+Be^{m_2t}$ where A and B are constants whose particular values may be determined by given additional boundary conditions.

If $\Delta=0$ then $m_1=m_2$ and the root is real and repeated. Then the solution to the original differential equation takes the form $x(t)=(A+Bt)e^{m_1t}$ where A and B are constants whose particular values may be determined by given additional boundary conditions.

If $\Delta<0$ the roots m_1 and m_2 are conjugate complex numbers where

$m_1=\alpha+i\beta$
$m_2=\alpha-i\beta$

Then the solution to the original differential equation takes the form

$x(t)=e^{\alpha t}(A\sin\beta t+B\cos\beta t)$

where A and B are constants whose particular values may be determined by given additional boundary conditions.

Inhomogeneous equations

A second order inhomogeneous, constant coefficient differential equation has the form

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+c=f(t)$$

where $f(t)\neq 0$. The function $f(t)$ is called the *forcing function*. To solve this type of equation we split the problem into two parts.

(1) We find the solution to the *complementary* homogeneous equation

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+c=0$$

This is called the *complementary function* and is denoted $x_c=x_c(t)$

(2) By **trial and error** (further illustrated below) we find the *particular* solution (a function) to the inhomogeneous equation, which is denoted $x_p=x_p(t)$

Then the general solution is

$$x(t)=x_c(t)+x_p(t)$$

In other words, we add the particular to the complementary function. We will now illustrate this method.

Example (1)

Find the general solution to $\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=2$

Solution

The homogeneous equation is

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=0$$

The auxiliary equation is

$$m^2+5m+6=0$$
$$(m+3)(m+2)=0$$
$$m_1=-3,\quad m_2=-2$$

Hence $x_c(t)=Ae^{-3t}+Be^{-2t}$

Now we have to find the particular solution. The forcing function is $f(t)=2$. It takes the form of a constant function $f(t)=k\ (k \text{ constant})$. To find the particular function, we will try $x_p(t)=k$ and deduce the value of k by trial and error.

Let $x_p(t)=k$. Then

$$\frac{d}{dt}x_p(t)=0, \quad \frac{d_2}{dt^2}x_p(t)=0$$

This function x_p must satisfy the differential equation. Hence, substituting

$\frac{d}{dt}x_p(t)=0, \quad \frac{d_2}{dt^2}x_p(t)=0$ into $\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=2$ we obtain

$$6k=2$$

$$k=\frac{2}{6}=\frac{1}{3}$$

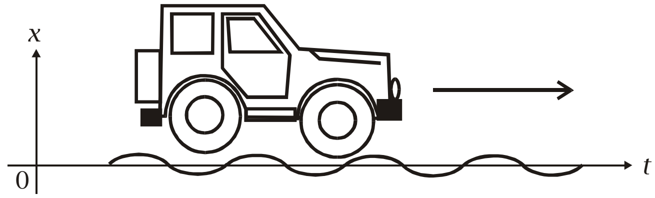
Hence $x_p(t)=\frac{1}{3}$.

Then the final solution is the sum of the complementary and particular functions

$$x(t)=x_c(t)+x_p(t)=Ae^{-3x}+Be^{-2x}+\frac{1}{3}$$

Interpretation

Now we will give a physical illustration of why adding a complementary to a particular function gives the general solution to this type of problem. Imagine a car driving over a bumpy road starting at the origin when time $t=0$ s.



The bumpy road has a form described by the function $f(t)$. The car's suspension system is modelled by a differential equation, which in this case will have an auxiliary equation with complex roots. The response to this will be a function describing how the car's undercarriage oscillates as a result of the motion over the bumpy road. Intuitively, this response depends on the particular nature of the road together with the response of the car's suspension system. On a level road, that is with $f(t)=0$, the *response* would be given by the solution to the homogeneous equation $x_c(t)$, in other words, it is purely a function of the car's suspension. But the road modifies this response and forces the car to follow the contours of the rough road. That is why the particular function $x_p(t)$ will take the same form as the forcing function. Thus the general solution is

$x(t) = x_c(t) + x_p(t)$

= natural response of the system + particular response due to the forcing function

Thus, when we seek $x_p(t)$ we use trial and error. But we narrow down the search by trying functions that have the same form as the forcing function.

Choosing a function to try

Hence, when seeking the particular solution to a differential equation of the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c = f(t)$$

we try functions **of the same form** as the forcing function $f(t)$.

Forcing function $f(t)$	**Form of particular function to try** $x_p(t)$
constant	$x_p(t) = k$
$k't + l'$	$x_p(t) = k + l$
$k'e^{\alpha' t}$	$x_p(t) = ke^{\alpha t}$
$k'\sin\theta' t + l'\cos\theta' t$	$x_p(t) = k\sin\theta t + l\cos\theta t$

Example (2)

Solve $2\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 3x = 3e^{-\frac{1}{2}t}$.

Solution

The homogeneous equation is

$$2\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 3x = 0$$

with auxiliary equation

$2m^2+5m-3=0$

$(2m-1)(m+3)=0$

$m_1=\frac{1}{2},\ \mathrm{m}_2=-3$

$x_c(t)=Ae^{\frac{1}{2}t}+Be^{-3t}$

For the particular solution, we try

$x_p(t)=ke^{-\frac{1}{2}t}$

$\frac{d}{dt}x_p=-\frac{1}{2}ke^{-\frac{1}{2}t} \quad \frac{d^2}{dt^2}x_p=\frac{1}{4}ke^{-\frac{1}{2}t}$

On substituting into the original equation

$\frac{k}{2}e^{-\frac{1}{2}t}-\frac{5k}{2}e^{-\frac{1}{2}t}-3ke^{-\frac{1}{2}t}=3e^{-\frac{1}{2}t}$

$\frac{k}{2}-\frac{5k}{2}-3k=3$

$-5k=5$

$k=-\frac{3}{5}$

Therefore

$$\begin{aligned} x(t)&=x_c(t)+x_p(t)\\ &=Ae^{\frac{1}{2}t}+Be^{-3t}-\frac{3}{5}e^{-\frac{1}{2}t}\end{aligned}$$

Example (3)

For the differential equation $\frac{d^2y}{dx^2}+9y=8e^{-x}$, find the solution for which $\frac{dy}{dx}=10$ and $y=2$ at $x=0$.

Solution

$\frac{d^2y}{dx^2}+9y=8e^{-x}$

The homogeneous equation is

$\frac{d^2y}{dx^2}+9y=0$

With auxiliary quadratic equation

$m^2+9=0$

$m^2=-9$

$m_{1,2} = \sqrt{-9} = \pm 3i$

The complementary function is

$y_c(x) = A\cos 3x + B\sin 3x$

where A and B are constants.

Let $y_p(x) = ke^{-x}$

$$\frac{d}{dx} y_p(x) = -ke^{-x}$$

$$\frac{d^2}{dx^2} y_p(x) = ke^{-x}$$

Substituting into the original equation

$ke^{-x} - 9ke^{-x} = 8e^{-x}$

$-8k = 8$

$k = -1$

$y_p(x) = -e^{-x}$

The general solution is

$y(x) = y_c(x) + y_p(x) = A\cos 3x + B\sin 3x - e^{-x}$

To find the coefficients A and B we use the boundary conditions. Firstly,

$y(0) = 2 \;\Rightarrow\; A - 1 = 2 \;\Rightarrow\; A = 3$

On differentiating

$$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x + e^{-x}$$

$$\frac{dy}{dx} = 10 \quad\Rightarrow\quad 3B + 1 = 10 \quad\Rightarrow\quad B = 3$$

Hence the particular solution is

$y(x) = 3\cos 3x + 3\sin 3x - e^{-x}$