Separation of variables by substitution

The need for integration by substitution

Recall the method of integration by substitution. For example, suppose we wish to integrate $\int \sec^2 x \left(e^{1+\tan x}\right) dx$. We proceed as follows, using the method of substitution

Let u $= 1 + \tan x$
Then $du = \sec^2 x dx$
and $dx = \dfrac{du}{\sec^2 x}$

Hence

$$\int \sec^2 x \left(e^{1+\tan x}\right) dx = \int \sec^2 x \times e^u \frac{1}{\sec^2 x} du$$
$$= u = e^u + c$$
$$= e^{1+\tan x} + c$$

The method of substitution is not in the *strictest* sense necessary. The integral, $\int \sec^2 x \cdot e^{1+\tan x} dx$, could be found by direct integration. The method of substitution is an aid to perception. The underlying structure of the integral is made clearer. The function $\sec^2 x$ is the derivative of $1 + \tan x$, and the substitution $u = 1 + \tan x$ makes this relationship clearer.

Substitutions to help solve differential equations by the separation of variables

We can also employ a form of the method of substitution when solving differential equations by the method of substitution. We shall demonstrate the technique by example.

Example (1)

Solve the equation $\dfrac{dy}{dx} = \dfrac{y - x + 3}{y - x - 6}$ by the change of variable $y - x = z$.

<u>Solution</u>

$$\frac{dy}{dx} = \frac{y - x + 3}{y - x - 6}$$

© blacksacademy.net

Let $z = y - x$ then $y = z + x$ and $\frac{dy}{dx} = \frac{dz}{dx} + 1$

On substituting

$$\frac{dz}{dx} + 1 = \frac{z+3}{z-6}$$

$$\frac{dz}{dx} = \frac{z+3}{z-6} - 1 = \frac{z+3-z+6}{z-6} = \frac{9}{z-6}$$

$$\int \frac{z-6}{9} dz = \int dx$$

Hence

$$\frac{1}{9}\left(\frac{z^2}{2} - 6z\right) = x + c$$

On substituting back into the equation

$$\frac{(y-x)^2}{2} - 6(y-x) = 9x + c$$

Let us look at a more advanced application of the same approach

Example (2)

Try to solve $\frac{dy}{dx} = \frac{x-y+4}{x+y}$ by the method of the separation of variables. Explain why you get stuck in the attempt.

Solution

We start with

$$\frac{dy}{dx} = \frac{x-y+4}{x+y}$$

The method of the separation of variables requires that we bring everything in y to one side of the equation and everything in x to the other side of the equation; but there does not appear to be any way of doing this. For instance

$$\int (x+y)dy = \int (x-y+4)dx$$

$$\int xdy + \int ydy = \int xdx - \int ydy + \int 4dx$$

But here we have reached an impasse. We have mixed expressions involving x and dy, for instance, and we cannot get any further.

So we require a different approach.

Example continued

Solve the equations $\frac{dy}{dx} = \frac{x - y + 4}{x + y}$, reducing it to a homogeneous equation by the change of variables $x = \tilde{x} - 2,\ y = \tilde{y} + 2$.

<u>Solution</u>

We have $\frac{dy}{dx} = \frac{x - y + 4}{x + y}$

Let $x = \tilde{x} - 2$ and $y = \tilde{y} + 2$

That is

$\tilde{x} = x + 2,\ \tilde{y} = y - 2$

So

$\frac{d\tilde{y}}{dt} = \frac{dy}{dt}$ and $\frac{d\tilde{x}}{dt} = \frac{dx}{dt}$

Hence

$\frac{dy}{dx} = \frac{d\tilde{y}}{d\tilde{x}}$

And so, on substituting into $(*)$

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{\tilde{x} - 2 - \tilde{y} - 2 + 4}{\tilde{x} - 2 + \tilde{y} + 2} = \frac{\tilde{x} - \tilde{y}}{\tilde{x} + \tilde{y}} = \frac{1 - \frac{\tilde{y}}{\tilde{x}}}{1 + \frac{\tilde{y}}{\tilde{x}}}$$

That is

$$\frac{d\tilde{y}}{d\tilde{x}} = \frac{1 - \frac{\tilde{y}}{\tilde{x}}}{1 + \frac{\tilde{y}}{\tilde{x}}} \qquad (**)$$

This has brought the equation to a new form where the expression $\frac{\tilde{y}}{\tilde{x}}$ appears on both the top and bottom of the fraction, and no where else. This suggests a further substitution will solve the problem.

Let $t = \frac{\tilde{y}}{\tilde{x}}$

then $\tilde{y} = t\tilde{x}$

and $\frac{d\tilde{y}}{d\tilde{x}} = \tilde{x}\frac{dt}{d\tilde{x}} + t\frac{d\tilde{x}}{d\tilde{x}} = \tilde{x}\frac{dt}{d\tilde{x}} + t$

On substituting into $\frac{d\tilde{y}}{d\tilde{x}} = \frac{1 - \frac{\tilde{y}}{\tilde{x}}}{1 + \frac{\tilde{y}}{\tilde{x}}}$ $(**)$ we get $t + \tilde{x}\frac{dt}{d\tilde{x}} = \frac{1 - t}{1 + t}$

© blacksacademy.net

Now, at last, we can separate the variables

$$\tilde{x}\frac{dt}{d\tilde{x}} = \frac{1-t}{1+t} - t = \frac{1-t-t-t^2}{1+t} = \frac{1-2t-t^2}{1+t}$$

$$\int \frac{1}{\tilde{x}} d\tilde{x} = \int \frac{1+t}{1-2t-t^2} dt = -\frac{1}{2}\int \frac{-2-2t}{1-2t-t^2} dt = -\frac{1}{2}\ln\left|1-2t-t^2\right|$$

We can now integrate

$$\ln\left|c\tilde{x}\right| = -\frac{1}{2}\ln\left|1-2t-t^2\right|$$

And on substituting back $t = \frac{\tilde{y}}{\tilde{x}}$, we obtain the solution to the original problem

$$\ln\left|c(x+2)\right| = -\frac{1}{2}\ln\left|1-2\frac{\tilde{y}}{\tilde{x}} - \left(\frac{\tilde{y}}{\tilde{x}}\right)^2\right|$$

Replacing $\tilde{x} = x+2,\ \tilde{y} = y-2$

$$\ln\left|c(x+2)\right| = -\frac{1}{2}\ln\left|1-2\times\frac{y+2}{x-2} - \left(\frac{y+2}{x-2}\right)^2\right|$$

$$\ln\left|c(x+2)\right| = \ln\left|1-2\times\frac{y+2}{x-2} - \left(\frac{y+2}{x-2}\right)^2\right|^{-\frac{1}{2}}$$

$$\ln\left|c(x+2)\right| = \ln\left(\frac{1}{\sqrt{\left|1-2\times\frac{y+2}{x-2} - \left(\frac{y+2}{x-2}\right)^2\right|}}\right)$$

We can remove the root sign by squaring both sides

$$\ln\left(c(x+2)\right)^2 = \ln\left(\frac{1}{1-2\times\frac{y+2}{x-2} - \left(\frac{y+2}{x-2}\right)^2}\right)$$

On taking the exponent of both sides, we obtain a final solution to the problem

$$c(x+2)^2 = \frac{1}{1-2\left(\frac{y+2}{x-2}\right) - \left(\frac{y+2}{x-2}\right)^2}$$

© blacksacademy.net