Sequences and Series

Sequences

A sequence is any string of numbers in a given order. It is usual to denote successive members of the sequence by letters with numerical subscripts:

$u_0, \; u_1, \; u_2, \; u_3, \; . . . . , u_n, \; . . . .$

u_n, means the nth number in the sequence.
The numbering of the subscripts is a matter of convention.
That is, the first term could be designated by u_1 or u_2 etc.

Often sequences are generated by a rule, for example,

$u_n = n^2$
which gives the sequence:
$u_0 = 0, \; u_1 = 1, \; u_2 = 4, \; u_3 = 9, \; . . . .$

A finite sequence can also be generated by a recursion formula. This is a rule for computing the n + 1 th term from previous terms. An example of this is the Fibonnacci sequence, given by the recursion formula:

$u_{n+1} = u_n + u_{n-1}$

When $u_0 = 1$ and $u_1 = 1$, this gives the sequence:

$u_0 = 1, \; u_1 = 1, \; u_2 = 2, \; u_3 = 3, \; u_4 = 5, \; u_5 = 8, \; u_6 = 13 \; . . . .$

A sequence can exhibit periodicity – meaning, it can repeat itself. A simple example is the alternating sequence:

$u_{n+1} = -u_n$
with $u_0 = 1$ this rule gives:

$u_0 = 1, \; u_1 = -1, \; u_2 = 1, \; u_3 = -1, \; . . . .$

This has period = 2. When the magnitude of the terms in a sequence gets larger and larger, the sequence is said to diverge, or be divergent. If the sequence tends towards a single value, then it is said to converge, or to be convergent.

Example (1)

Write down the first four terms and the n th term of the sequence whose r th term is:

$$\frac{r}{2r+1}$$

Solution

$a_r = \frac{r}{2r+1}$

$a_1 = \frac{1}{2\times 1+1} = \frac{1}{3}$

$a_2 = \frac{2}{2\times 2+1} = \frac{2}{5}$

$a_4 = \frac{3}{2\times 3+1} = \frac{3}{7}$

$a_4 = \frac{4}{2\times 4+1} = \frac{4}{9}$

The nth term is

$a_n = \frac{n}{2n+1}$

Example (2)

For the sequence $u_1, u_2, u_3, \ldots, u_n, \ldots$ the terms are related by $u_n = 3u_{n-2} + 4u_{n-1}$ where $n \geq 1$, $u_1 = 1$ and $u_2 = 2$. Find the value of u_3, u_4 and u_5 State whether this sequence is convergent or divergent.

Solution

$u_n = 3u_{n-2} + 4u_{n-1}$

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 2 \\ u_3 &= 3u_1 + 4u_2 \\ &= 3\times 1 + 4\times 2 = 11 \\ u_4 &= 3u_2 + 4u_3 \\ &= 3\times 2 + 4\times 11 = 50 \\ u_5 &= 3u_3 + 4u_4 \\ &= 3\times 11 + 4\times 50 = 233 \end{aligned}$$

Clearly the sequence is divergent.

Example (3)

The sequence u_1, u_2, u_3, … is defined by $u_n = 4n^2$.

(i) Write down the value of u_5.

(ii) Express $u_{n+1} - u_n$ in terms of n, simplifying your answer.

(iii) The differences between successive terms of the sequence form another sequence in which the successive terms differ from each other by a common difference. Find the first term of this sequence, and its common difference. Prove that the common difference of this sequence is constant.

<u>Solution</u>

(i) $u_5 = 4 \times 5^2 = 4 \times 25 = 100$

(ii) $u_{n+1} - u_n = 4(n+1)^2 - 4n^2$

$= 4(n^2 + 2n + 1) - 4n^2$

$= 4n^2 + 8n + 4 - 4n^2$

$= 8n + 4$

$= 4(2n+1)$

(*iii*) Let $a_n = u_{n+1} - u_n = 4(2n+1)$

Then

$a_1 = 4(2+1) = 12$

$a_2 = 4(4+1) = 20$

$a_3 = 4(6+1) = 28$

Thus the first term is

$a_1 = 12$

And the common difference is

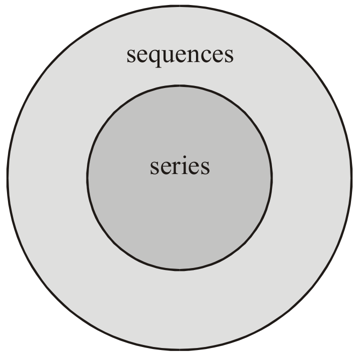
$d = 8$

To prove that the common difference is constant

$$a_n - a_{n-1} = 4(2n+1) - 4(2(n-1)+1)$$
$$= 8n + 4 - 4(2n - 2 + 1)$$
$$= 8n + 4 - 8n + 4$$
$$= 8$$

Series

Series are special types of sequences. They are formed from existing sequences by the addition of succeeding terms:



All series are sequences, but not all sequences are series.

To state the rule for the formation of a series we use the Σ notation.

Σ means "sum"

$\sum_{i=0}^{n} u_i$ means "sum the squence whosefirst term is u_0 and last term is u_n"

that is:

$$\sum_{i=0}^{n} u_i = u_0 + u_1 + u_2 + \ldots\ldots + u_n.$$

The u_i represents the ith term in the sequence

A series is generated by a rule of the form:

$$a_r = \sum_{i=0}^{r} u_i$$

This means:

$$a_0 = u_0$$
$$a_1 = u_0 + u_1$$
$$a_2 = u_0 + u_1 + u_2$$
$$\vdots$$
$$a_r = u_0 + u_1 + u_2 + \ldots + u_r.$$

Example (4)

Write down the first 4 terms of the series $\sum_{r=1}^{n}(2r+1)$

Solution

$a_n = \sum_{r=1}^{n}(2r+1)$

$a_1 = (2\times 1+1) = 3$
$a_2 = a_1 + (2\times 2+1) = 3+5 = 8$

$a_3 = a_2 + (2\times 3+1) = 8+7 = 15$
$a_4 = a_3 + (2\times 4+1) = 15+9 = 24$