Series Expansion

Functions can be multiplied together. Suppose two functions $f(x)$ and $g(x)$ each have a series expansion about a common point.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + ...$$

Then the series expansion of the product of these two functions is found by multiplying the two series together. This has to be done systematically on a term by term basis.

$$f(x) \times g(x) = \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...\right)\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + ...\right)$$
$$a_0 b_0 + \left(a_0 b_1 + a_1 b_0\right)x + \left(a_0 b_2 + a_1 b_1 + a_2 b_0\right)x^2 + ...$$

So we can find the series expansion of a product of functions by using our knowledge of algebra in a very intuitive way. This is best illustrated by example

Example

Express $f(x) = \dfrac{1-x}{1+2x}$ as a series of ascending powers of x, as far as the term is x^3, giving the values x for which the series converges to f (x).

Solution

$$f(x) = \frac{1-x}{1+2x}$$

We begin by writing this as a product of two functions

$$\frac{1-x}{1+2x} = (1-x)(1+2x)^{-1}$$

The first of these functions does not require expansion. The second can be expanded by the binomial theorem with negative index. For

$$(1+2x)^{-1} = 1 + \frac{(-1)}{1!}(2x) + \frac{(-1)(-2)}{2!}(2x)^2 + \frac{(-1)(-2)(-3)}{3!}(2x)^3 + ...$$

Hence

$$\frac{1-x}{1+2x} = (1-x)(1+2x)^{-1}$$
$$= (1-x)(1-2x+4x^2-8x^3+\dots)$$

Now we multiply out the pair of brackets on a term by term basis, but remembering that the second bracket contains an infinite number of terms.

Hence

$$\frac{1-x}{1+2x} = 1-2x+4x^2-x+2x^2-\dots$$
$$= 1-2x+4x^2-8x^3+\dots$$
$$= 1-3x+6x^2-12x^3+\dots$$

The series converges to $f(x)$ if $|2x|<1$

That is

$|x|<\frac{1}{2}$