Simple Harmonic Motion

Prerequisites

You should already be familiar with elementary differential and integral calculus and the formation of a differential equation. You should also be familiar with the concept of angular velocity, from your study of motion in a horizontal circle or otherwise. Study the following example carefully.

Example (1)

(*a*) (*i*) Differentiate twice $x = A\sin(\omega t)$, where A is a constant. Hence find $\frac{d^2x}{dt^2}$ and show that $x = A\sin\omega t$ is a solution to the differential equation $\frac{d^2x}{dt^2} = -\omega^2 x$.

(*ii*) Show also that $x = A\cos(\omega t)$ (A, constant) is a solution to the differential equation $\frac{d^2x}{dt^2} = -\omega^2 x$.

(*iii*) What are the values of $x = A\sin(\omega t)$ and $x = A\cos(\omega t)$ when $t = 0$?

(*iv*) The motion of a particle is given by the equation

$x = A\sin(\omega t)$

At time $t = 0$ the particle is at the origin, and has velocity $6\ \text{ms}^{-1}$. Find *A*.

(*v*) The motion of a particle is given by the equation

$x = A\cos(2t)$

If its acceleration at $t = 0$ is $\frac{d^2x}{dt^2} = -2\ \text{ms}^{-2}$ find the value of *A*, and state also the displacement of the paticle when $t = 0$.

(*b*) A particle is moving in a straight line so that its acceleration $(a\ \text{ms}^{-2})$ towards a fixed point *O* is proportional to its distance $(x\ \text{m})$ from *O* and directed towards *O*. When the particle is at *O* its displacement is 0 and its velocity is $6\ \text{ms}^{-1}$. When it is 2 m from *O* its acceleration is $-8\ \text{ms}^{-2}$. Form a differential equation and solve it to find the equation governing the motion of the particle.

Solution

(a) (i) $x = A\sin(\omega t)$

Differentiating this twice

$$\frac{dx}{dt} = A\omega\cos(\omega t)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t) \qquad (1)$$

On substituting $x = A\sin(\omega t)$ into (1)

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Hence $x = A\sin(\omega t)$ is a solution to the differential equation $\frac{d^2x}{dt^2} = -\omega^2 x$

(ii) $x = A\cos(\omega t)$

$$\frac{dx}{dt} = -A\omega\sin(\omega t)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t) \qquad (1)$$

Hence $x = A\cos(\omega t)$ is also a solution to $\frac{d^2x}{dt^2} = -\omega^2 x$

(iii) $x = A\sin(\omega t)$

When $t = 0,\ x = A\sin(0) = 0$

$x = A\cos(\omega t)$

When $t = 0,\ x = A\cos(0) = A$

(iv) $x = A\sin(3t)$

This particle starts at the origin, and its velocity is

$$v = \frac{dx}{dt} = 3A\cos(3t)$$

Hence, on substituting $v = 6,\ t = 0$

$6 = 3A$

$A = 2$

(v) $x = A\cos(2t)$

This particle starts at A, and its velocity is

$$v = \frac{dx}{dt} = -2A\sin(2t)$$

Its acceleration is

$$a = \frac{d^2x}{dt^2} = -4A\cos(2t)$$

Hence, on substituting $a = -2,\ t = 0$

$-2 = -4A$

$A = 0.5$

This gives the initial displacement of the particle at $t = 0$. Hence, initial displacement = 0.5 m (*b*) The phrase, "a particle is moving in a straight line so that its acceleration $\left(a \text{ ms}^{-2}\right)$ towards a fixed point *O* is proportional to its distance $\left(x \text{ m}\right)$ from *O* and directed towards *O*" translates into the statement

$a \propto -x$

$a = -kx \qquad k$ constant

To turn this into a differential equation we observe that acceleration is the second derivative of displacement, hence

$a = \dfrac{d^2x}{dt^2}$ and we have $\dfrac{d^2x}{dt^2} = -kx \qquad k$ constant

In this question we are given additional information about the acceleration and velocity of the particle at various times or positions. We call such information *boundary conditions* or *initial conditions*. The boundary conditions given in this question are

(*i*) At $x = 2,\ a = \dfrac{d^2x}{dt^2} = -8$

(*ii*) At $t = 0,\ x = 0$ and $v = 6$

where *x, a* and *v* stand for displacement, acceleration and velocity. So, substituting the boundary condition, $x = 2,\ a = \dfrac{d^2x}{dt^2} = -8$

$-8 = -k \times 2 \qquad \Rightarrow \qquad k = 4$

Hence, we have the differential equation

$$\frac{d^2x}{dt^2} = -4x$$

In part (*a*) we saw that either $x = A\sin\left(\omega t\right)$ or $x = A\cos\left(\omega t\right)$ is a solution to the equation $\dfrac{d^2x}{dt^2} = -\omega^2 x$. However, in the question we are given that the displacement of the particle at $t = 0$ is $A = 0$, so the equation that we should use is $x = A\sin\left(\omega t\right)$. Its second derivative gives the acceleration of the particle as

$$a = \frac{d^2x}{dt^2} = -A\omega^2 \sin\left(\omega t\right)$$

By comparing this with

$$\frac{d^2x}{dt^2} = -4x = -4A\sin\left(\omega t\right)$$

we see that $\omega = \sqrt{4} = 2$ giving $x = A\sin\left(2t\right)$. To find the constant *A* in this first observe that when $t = 0$ then $x = A\sin\left(0\right) = 0$; hence we use the second boundary condition, that $t = 0,\ v = \dfrac{dx}{dt} = 6$. That is

$$x = A\sin(2t)$$
$$\frac{dx}{dt} = 2A\cos(2t)$$
$$t = 0, \frac{dx}{dt} = 6 \quad \Rightarrow \quad 6 = 2A \quad \Rightarrow \quad A = 3$$
Hence
$$x = 3\sin(2t)$$

The differential equation $\frac{d^2x}{dt^2} = -\omega^2 x$ is the equation of the *simple harmonic oscillator*, and its solution is a sine function, either $x = A\sin(\omega t)$ or $x = A\cos(\omega t)$. Any particle whose motion is governed by these two equations is said to undergo *simple harmonic motion*.

Which solution to choose?

To solve the equation $\frac{d^2x}{dt^2} = -\omega^2 x$ you must choose one of the solutions

$x = A\sin(\omega t)$ or $x = A\cos(\omega t)$.

This choice depends on the boundary conditions given in the question. If you are told that the displacement at time $t = 0$ is $x = 0$, then you choose the equation $x = A\sin(\omega t)$. If you are told that the displacement at $t = 0$ is $x = A$, then you choose the equation $x = A\cos(\omega t)$.

In fact, in addition to $x = A\sin(\omega t)$ or $x = A\cos(\omega t)$ as solutions to $\frac{d^2x}{dt^2} = -\omega^2 x$, any linear combination of these two is also a potential solution. That is, $x = A\sin(\omega t) + B\cos(\omega t)$ is also a solution to $\frac{d^2x}{dt^2} = -\omega^2 x$. In this chapter we restrict our attention only to solutions of the form $x = A\sin(\omega t)$ or $x = A\cos(\omega t)$ and the given boundary conditions will require you to choose between them.

Remark

It is worth remarking that the solution to example (1) is based on techniques that you should already know. Yet, using these techniques we have already been able to construct almost the entire background theory of simple harmonic motion, as we shall proceed to demonstrate in further detail.

Simple harmonic motion

Simple harmonic motion

Simple harmonic motion is defined as motion taking place along a straight-line in which the acceleration of the object is proportional to the displacement of the object from a fixed point and in the opposite direction to the displacement.

For simple harmonic motion to occur there must be

(1) A mass that oscillates

(2) A central point of equilibrium about which the mass oscillates

(3) A restoring force that pulls the mass towards the point of equilibrium. The restoring force acts in a direction that is opposite to the displacement of the mass and is proportional to that displacement

The differential equation governing simple harmonic motion is

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where a is the acceleration, x is the displacement, and ω is the angular velocity. It has solution

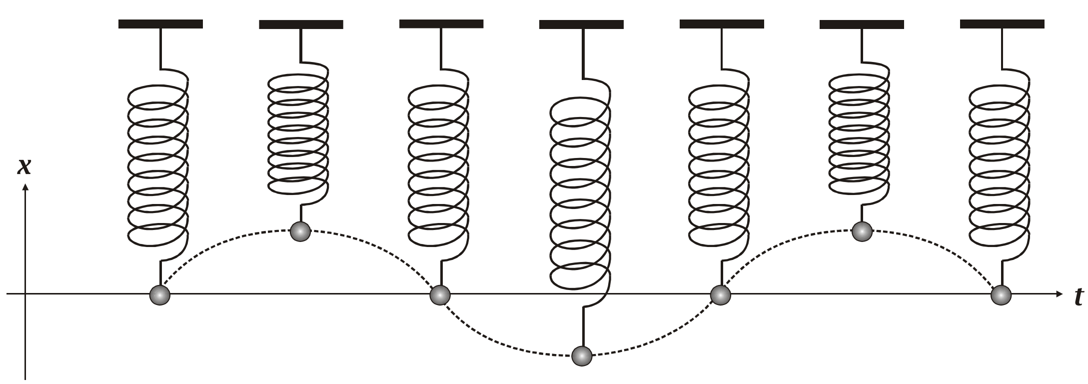
$x = A\sin\omega t$ or $x = A\cos\omega t$

where A is the amplitude.

In the above we state that the term ω in the differential equation $\frac{d^2x}{dt^2} = -\omega^2 x$ is called the *angular velocity* of the simple harmonic motion. This will be explained in more detail below.

Physical applications

A mass attached to the end of a helical spring or elastic string will exhibit simple harmonic motion. As the spring alternately contracts and expands the motion of the bob over time describes a sine wave.



If air resistance and other sources of loss of energy can be ignored then such a physical system would oscillate indefinitely. In practice, friction and energy loses cannot always be ignored; nonetheless, examples of physical situations that exhibit behaviour that is either simple harmonic motion or akin to it include

Vibrating strings of a violin	Alternating currents in a television aerial
Sound waves in a pipe	Electromagnetic waves
Vibrations of machinery	Vibrations of an atom
Alternating current in an electrical circuit	Vibrations of crystal structures

Physical systems that exhibit harmonic motion are called simple *harmonic oscillators*.

Oscillations

In this subsection we aim to gain more insight into the nature of solutions to the differential equation of the harmonic oscillator

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

We shall restrict our attention to the solution that takes the form $x = A\sin\omega t$. The function $x = A\sin\omega t$ describes the motion of an oscillating mass. It is an example of a *waveform* and has several properties.

The *displacement* (x) is the distance of the mass from the fixed point O to which the restoring force is directed.

The *amplitude* (A) is the maximum displacement.

The *period* (T) is the time taken for the mass to complete one whole cycle of displacements in both directions from *O*.

The *frequency* (f) is the number of cycles (periods) per unit of time; that is, the number of cycles per second.

The oscillating mass has also the properties of *velocity* $\left(v = \frac{dx}{dt}\right)$ and *acceleration* $\left(a = \frac{dv}{dt} = \frac{d^2x}{dt^2}\right)$

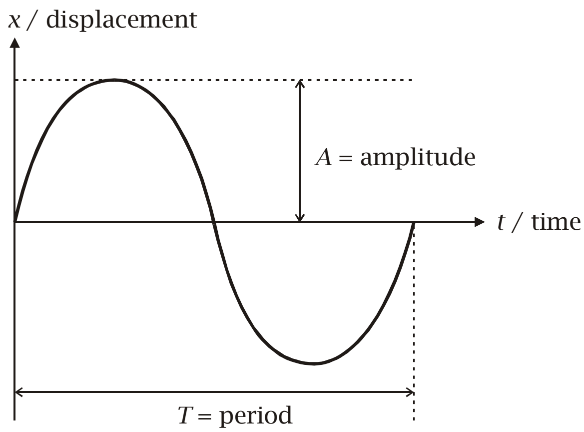
The frequency (f) and period (T) are not directly expressed in the equation

$$x = A\sin\omega t$$

We shall relate both to the angular velocity (ω), which is another property of this waveform, that we shall define below. Period and frequency are related to each other, as frequency is the reciprocal of period.

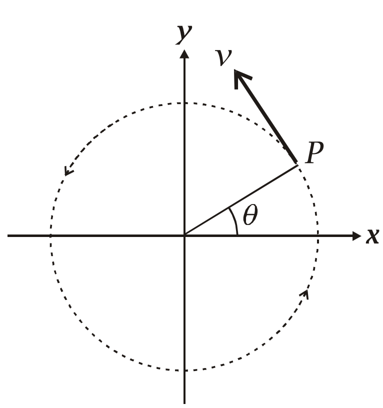
$f = \frac{1}{T}$ frequency $= \frac{1}{\text{period}}$

The standard units of period are seconds; those of frequency are hertz (symbol, Hz). I hertz is a frequency of 1 cycle per second.



Angular velocity

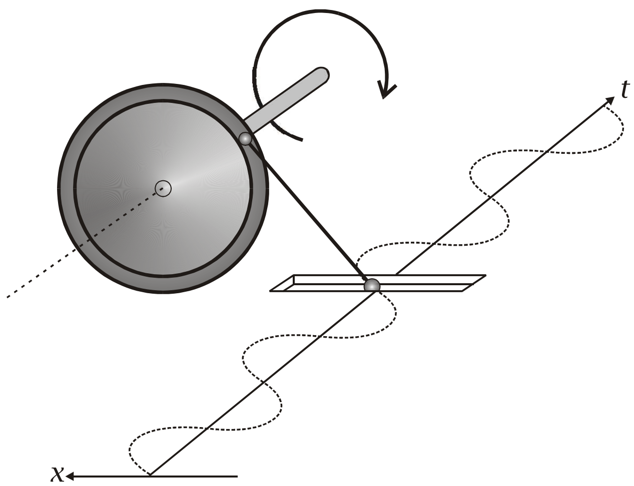
You should have already met the concept of angular velocity in the context of a particle moving in a horizontal circle. Let a particle P be moving at a constant speed v ms^{-1} in a horizontal circle. Let the origin of the motion be placed at the centre of the circle and let us also establish a set of coordinates for the horizontal plane in which the motion takes place.



Let the angle made by the particle and the *x*-axis be θ radians. As *P* moves around the circle the angle θ increases. Suppose at time $t = 0$ s the angle is also $\theta = 0$ so that we fix the direction of the *x*-axis by this means. Then clearly as *t* increases the angle θ increases. We call the angle swept out per unit of time *angular velocity* and denote this by ω.

$$\text{angular velocity} = \frac{\text{change in angle}}{\text{change in time}} \qquad \omega = \frac{\theta}{t}$$

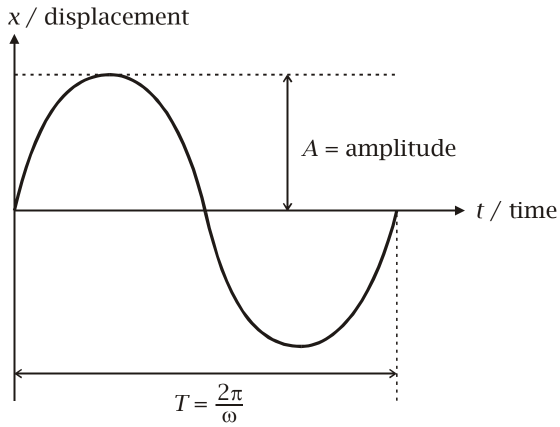
We remind you that the units of angular velocity are radians per second, rad s^{-1}. We shall now see that we can relate this concept of angular velocity to the period and frequency of a harmonic oscillator. The following diagram illustrates another object that describes simple harmonic motion.



The wheel is turning with a constant angular velocity. A shaft is attached to the wheel at one end. At the other end of the shaft a ball is fixed. As the wheel turns the ball is constrained to move within a groove. The resultant oscillations of the ball in the groove describe a sine wave. The period of one complete cycle corresponds to a single revolution of 2π radians. If the frequency is *f* Hz then the period is $\frac{1}{f}$ seconds. So in $\frac{1}{f}$ seconds the curve sweeps out 2π radians. Hence, in 1 second the curve sweeps out $2\pi f$ radians. But the angle swept out per unit time is the same as angular velocity (ω). That is, the angular velocity of the oscillation is $2\pi f \text{ rad s}^{-1}$ where *f* is the frequency. Thus

$$\omega = 2\pi f$$

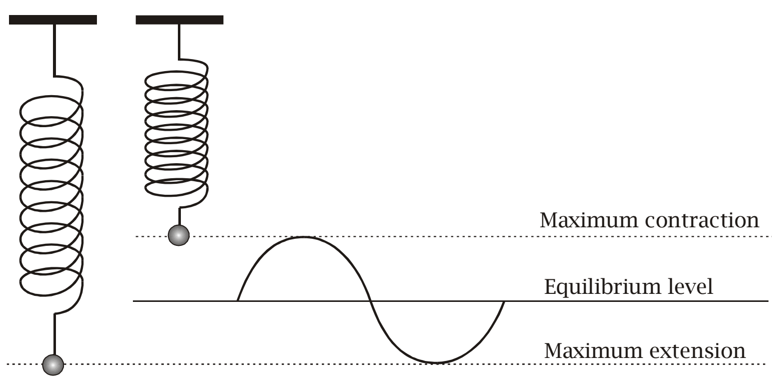
Further, since $f = \frac{1}{T}$ we also have $\omega = \frac{2\pi}{T}$ and $T = \frac{2\pi}{\omega}$



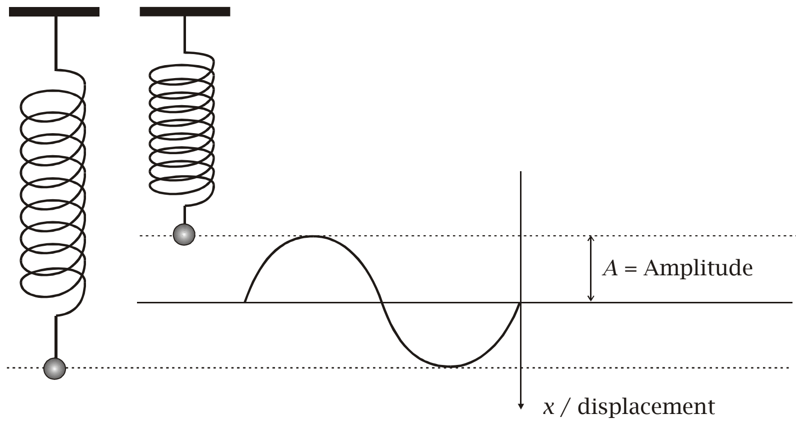
Note in this graph that we have indicated that the period of the oscillation is $T = \frac{2\pi}{\omega}$ seconds. In all of the above angles are measured in radians. This is standard for problems set on harmonic oscillators. However, if angles are measured in degrees then the period is $T = \frac{360}{\omega}$. So it is important to be clear as to whether angles are measured in degrees or radians.

The physics of simple harmonic motion

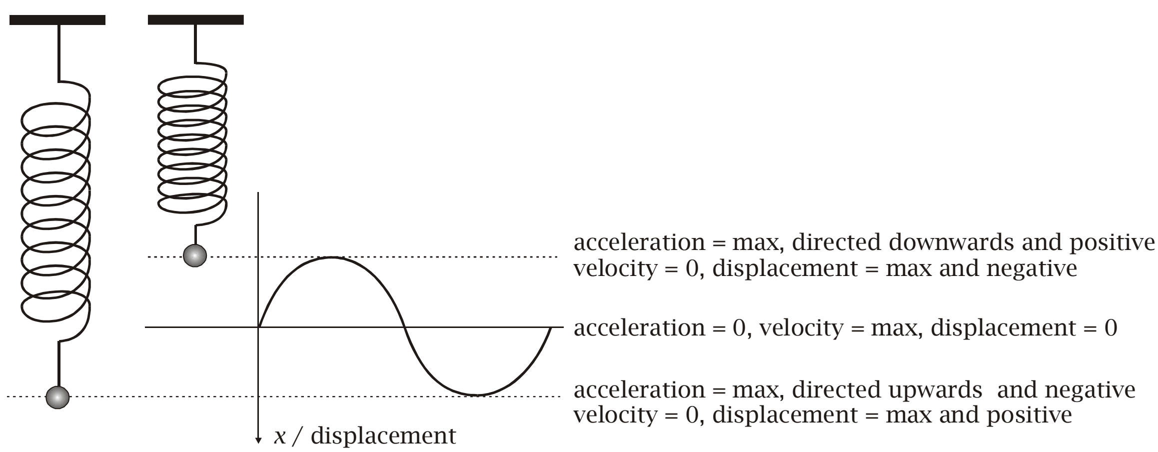
When a spring oscillates it does so about an equilibrium level.



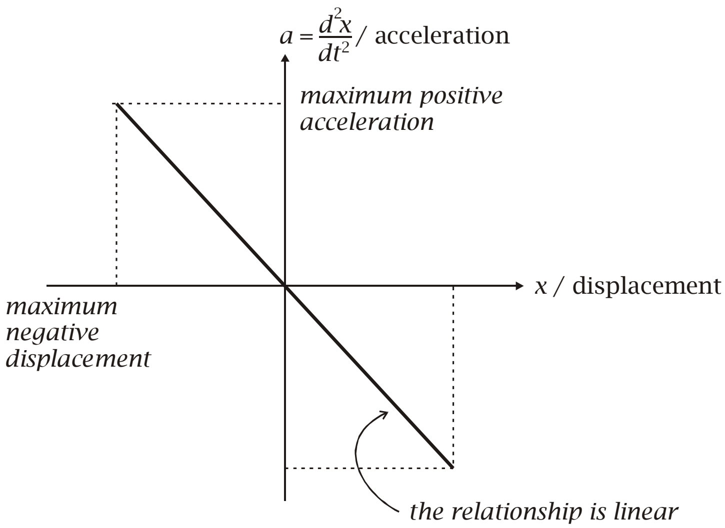
There is a maximum contraction and a maximum extension. In this physical interpretation of simple harmonic motion, we measure the displacement as the distance travelled by the mass downwards from the equilibrium level. The amplitude is the maximum displacement in both a positive and negative direction.



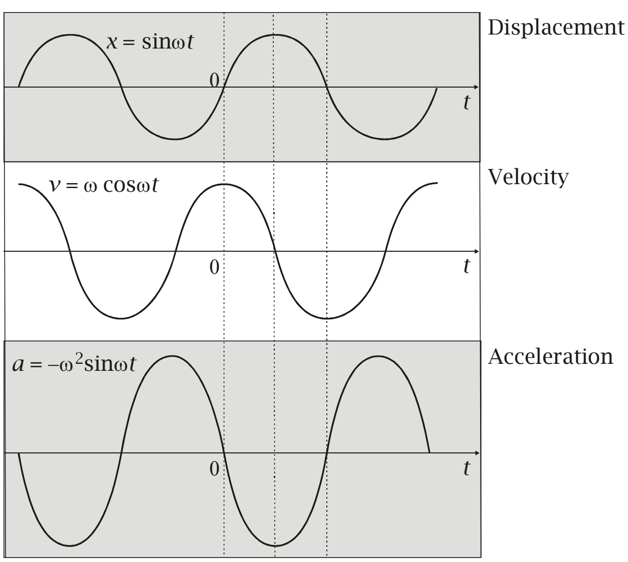
When the spring is contracting and the mass lies above the equilibrium level, it has a negative displacement. At the two points of maximum amplitude (maximum contraction and maximum extension), the weight has no velocity. On the other hand, its acceleration is at a maximum, because it is just changing direction and the size of the restoring force is proportional to the displacement. In this physical application at both points the spring is exerting the maximum force to "pull" the weight back up, or to "push" the weight back down towards the equilibrium level. As the weight passes through the equilibrium level it has maximum velocity, but the spring is not exerting a force on the weight at all and its acceleration is, therefore, zero.



The diagram illustrates how the acceleration and velocity of the mass changes as it oscillates. Simple harmonic motion is defined by the linear relationship between acceleration and displacement. We can now illustrate this graphically.



The displacement/time graph of an object in simple harmonic motion takes the form of a sine wave. The velocity of an object in simple harmonic motion is its gradient. Therefore, when displacement is at a maximum or minimum, the velocity is 0 – since the gradient of a maximum or minimum is 0. When displacement is 0, the velocity is at a maximum. Since $\cos(\omega t) = \sin\left(\omega t + \frac{\pi}{2}\right)$ the velocity is $\frac{\pi}{2}$ ahead of the displacement. Acceleration is the gradient of the velocity/time graph. Acceleration is proportional but in the opposite direction to the displacement. When displacement has a maximum, acceleration is at a minimum, and vice-versa. This gives the following graphs



Relationships in harmonic motion

We can describe several relationships between the various variables and constants that we have defined.

Relationship between velocity, angular frequency, displacement and amplitude

$$v^2 = \omega^2\left(A^2 - x^2\right)$$

where v is the velocity, ω is the angular frequency, A is the amplitude and x is the displacement.

Proof

Let $x = A\sin\omega t$, then

$$v = \frac{dx}{dt} = A\omega\cos\left(\omega t\right)$$

Hence

$$\begin{aligned} v^2 &= A^2\omega^2\cos^2\left(\omega t\right) \\ &= A^2\omega^2\sqrt{1-\sin^2\left(\omega t\right)} \\ &= \omega^2\sqrt{A^2 - A^2\sin^2\left(\omega t\right)} \\ &= \omega^2\left(A^2 - x^2\right) \end{aligned}$$

A similar proof would also apply to the form $x = A\cos\omega t$.

Relationship between maximum velocity and angular frequency

$$v_{\max} = \omega A$$

where A is the amplitude and ω is the angular frequency and $v_{\max}$ denotes the magnitude of the maximum velocity $v_{\max} = \left|v_{\max}\right|$.

Proof

We have $v^2 = \omega^2\left(A^2 - x^2\right)$. At the maximum velocity $x = 0$; hence

$$\begin{aligned} {v_{\max}}^2 &= \omega^2 A^2 \\ \left|v_{\max}\right| &= \omega A \end{aligned}$$

Maximum acceleration

The magnitude of the maximum acceleration is given by

$$\left|a\right| = \omega^2 A$$

where a is the acceleration, ω is the angular frequency and A is the amplitude.

Proof

Let $x = A\sin(\omega t)$ and $a = \frac{d^2x}{dt^2} = -A\omega^2 \sin(\omega t)$.

Hence, at the maximum acceleration $t = nT = \frac{2n\pi}{\omega}$ $\quad n = 0, \pm 1, \pm 2, \ldots$ and

$$a_{\max} = -A\omega^2 \sin(2n\pi) = \pm A\omega^2$$
$$|a| = A\omega^2$$

Example (2)

An object is oscillating under simple harmonic motion with amplitude 0.03 m and a frequency of 30 Hz. Find

(*a*) The period of oscillation

(*b*) The maximum acceleration

(*c*) The maximum velocity.

Solution

$A = 0.03$ m

$f = 30$ Hz

(*a*) $T = \frac{1}{f} = \frac{1}{30} = 0.033 \text{ s} \quad (2 \text{ s.f.})$

(*b*) $\omega = 2\pi f = 2 \times \pi \times 30 = 60\pi \text{ rads}^{-1}$

$a = -\omega^2 x$

$\therefore$ the magnitude of the maximum acceleration is given by

$a_{\max} = \left|-(60\pi)^2 \times 0.03\right| = 108\pi^2 \text{ ms}^{-2}$

The minimum acceleration occurs at the middle of the oscillation, and is zero

(*c*) $v_{\max} = \omega \times A = 60\pi \times 0.03 = 1.8\pi \text{ ms}^{-1}$

Example (3)

A particle *P* is moving in a straight line with simple harmonic motion about centre *O*, with frequency $\frac{1}{6}$ Hz. The maximum speed of *P* is $4\pi \text{ ms}^{-1}$.

(*a*) Find the amplitude of the motion.

(*b*) At a point *A* the velocity of *P* is $2\pi \text{ ms}^{-1}$. Find the distance *OA*, expressing your answer in surd form.

(*c*) Calculate the time taken by *P* to move directly from *O* to *A*.

(*d*) Determine the magnitude of the maximum acceleration of *P*.

(*e*) Find the distance travelled by the particle *P* during the first 30 s of its motion.

Solution

(*a*) $\omega = 2\pi f = 2\pi \times \frac{1}{6} = \frac{\pi}{3}$

$$v_{\max} = \omega A$$
$$4\pi = \frac{\pi}{3} A$$
$$A = 12 \text{ m}$$

(*b*) $v^2 = \omega^2 \left(A^2 - x^2\right)$

$$v = -2\pi,\ A = 12,\ \omega = \frac{\pi}{3}$$
$$4\pi^2 = \frac{\pi^2}{9}\left(144 - x^2\right)$$
$$144 - x^2 = 36$$
$$x^2 = 144 - 36 = 108$$
$$x = \sqrt{108} = 6\sqrt{3} \text{ ms}^{-1} \text{ (3 s.f.)}$$

(*c*) The equation governing the motion of *P* is

$$x = 12\sin\left(\frac{\pi}{3}t\right)$$

We choose this equation because it places the position of the particle at $t = 0$ at the origin, *O*, $x = 0$. Then, if the particle moves directly from *O* to *A* then we may substitute $x = 6\sqrt{3}$ into this equation to obtain

$$6\sqrt{3} = 12\sin\left(\frac{\pi}{3}t\right)$$
$$\sin\left(\frac{\pi}{3}t\right) = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$$
$$\frac{\pi}{3}t = \frac{\pi}{3}$$
$$t = 1 \text{ s}$$

(*d*) $|a| = A\omega^2 = 12 \times \left(\frac{\pi}{3}\right)^2 = \frac{4\pi^2}{3} = 13.2 \text{ ms}^{-2} \text{ (3 s.f.)}$

(*e*) The period is $T = \frac{1}{f} = \frac{1}{\left(\frac{1}{6}\right)} = 6 \text{ s}$

Therefore in 30 s the particle makes $30 \div 6 = 5$ periods. The distance travelled in any one period is 4 times the amplitude. Therefore the distance travelled in 30 s is $d = 5 \times 4 \times 12 = 240 \text{ m}$.

Summary

t is time

x is the displacement

v is the velocity

a is acceleration

f is frequency

T is period

ω is the angular frequency $\left(\text{rad s}^{-1}\right)$

A is the amplitude

Simple harmonic motion is defined as motion taking place along a straight-line in which the acceleration of the object is proportional to the displacement of the object from a fixed point and in the opposite direction to the displacement. The differential equation governing simple harmonic motion is $\frac{d^2x}{dt^2} = -\omega^2 x$ with the following solution.

$x = A\sin\omega t$

with boundary condition

$x = 0$ at $t = 0$

Then

$$v = \frac{dx}{dt} = A\omega\cos(\omega t)$$

$$a = \frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t)$$

$x = A\cos\omega t$

with boundary condition

$x = A$ at $t = 0$

Then

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t)$$

$$a = \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t)$$

The following relationships hold

$$f = \frac{1}{T}$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$T = \frac{2\pi}{\omega}$$

$$v^2 = \omega^2\left(A^2 - x^2\right)$$

$$\left|v_{\max}\right| = \omega A$$

$$\left|a_{\max}\right| = \omega^2 A$$