Simple Harmonic Motion and Springs

Prerequisites

You should already be familiar with simple harmonic motion and with Hooke's law for springs and elastic materials.

Simple harmonic motion

t is time	f is frequency
x is the displacement	T is period
v is the velocity	ω is the angular frequency
a is acceleration	A is the amplitude

Simple harmonic motion is defined as motion taking place along a straight-line in which the acceleration of the object is proportional to the displacement of the object from a fixed point and in the opposite direction to the displacement. The differential equation governing simple harmonic motion is $\frac{d^2x}{dt^2} = -\omega^2 x$ with solution

$$x = A\sin\omega t \qquad v = \frac{dx}{dt} = A\omega\cos(\omega t) \qquad a = \frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t)$$

The following relationships hold

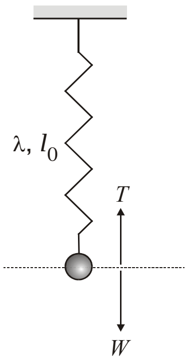
$$f = \frac{1}{T} \qquad v^2 = \omega^2\left(A^2 - x^2\right)$$

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad |v_{\max}| = \omega A$$

$$T = \frac{2\pi}{\omega} \qquad |a_{\max}| = \omega^2 A$$

Springs are harmonic oscillators

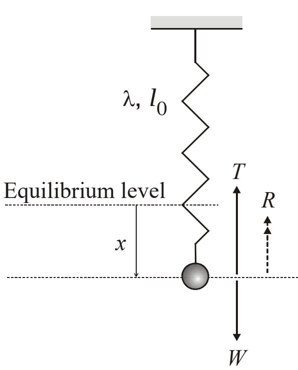
We will now show that a spring or elastic string suspended vertically, with a mass attached at the lower end and the higher end attached to a fixed point is a harmonic oscillator. We firstly consider a spring hanging in equilibrium (that is, not moving) under the weight of the mass attached to it.



The spring has natural length l_0 and modulus of elasticity λ. Its stiffness (or spring constant) k is given by $k = \frac{\lambda}{l_0}$. There are two forces acting on the mass. These are the tension produced by the spring, and its weight. The spring is not oscillating, so these two forces are equal, $T = W$. The tension is produced by the extension of the spring beyond its natural length. Let this extension be *d*. Then, by Hooke's law $T = kd$ where k is the stiffness of the spring. Hence, since $T = W$ we have

$kd = mg$

This equation will be used below as we proceed to show that once the mass is disturbed from its equilibrium position it will undergo simple harmonic motion. So now imagine that the mass is pulled down below the equilibrium point, thus increasing the tension above the weight. This means that if the mass is released it will be pulled upwards by the resultant force. We claim that the mass will start to oscillate. Let the extension *beyond* the equilibrium level be *x*.



The resultant force acts upwards initially – that is, in the opposite direction to the direction of the displacement.

Here we assume that there is no air resistance, so the magnitude of this resultant force is

$|R| = T - W$

The total extension of the spring is $y = x + d$, where d is, as before, the extension up to the equilibrium point, and x is the additional extension. Since $W = mg$, we have

$|R| = k(x + d) - mg$

But this is the *magnitude* of the resultant force, and not its direction. As already indicated the resultant acts in the opposite direction to the displacement, so to represent the direction we introduce a negative sign

$R = -\left(k\left(d + x\right) - mg\right)$

$R = -kd - kx + mg$

But we showed earlier that $mg = kd$. Hence

$R = -kd - kx + kd$
$R = -kx$

R is the resultant force acting on the mass and obeys Newton's second law, $F = ma$. Hence

$ma = -kx$

where $a = \frac{d^2x}{dx^2}$ is the acceleration of the mass. Hence

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$

This conforms to the definition of a simple harmonic oscillator (see introductory section). Hence, a mass attached to a spring, or light elastic string, is a simple harmonic oscillator. The angular frequency of the oscillation is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\lambda}{ml_0}}$$

where λ is Young's modulus for the spring, and l_0 is its natural length.

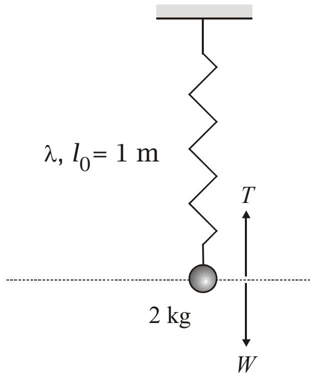
Example (1)

(*a*) A particle of mass 2 kg is attached to one end of a light elastic string *AB* of natural length 1 m, where *A* is fixed and *B* is the point of attachment between the particle and the string. The string is extended by a length of 0.24 m and the mass is in static equilibrium. Determine the modulus of elasticity, giving your answer as a fraction.

(*b*) The particle is pulled down a further 0.3 m after which it is released from rest at time $t = 0$. Show that its subsequent motion is harmonic. Find the time after which the particle passes through the equilibrium position for the first time and its speed when it does so.

Solution

(*a*)



At equilibrium $T = W$. The tension is given by $T = kd$ where k is the spring constant and d is the extension. The spring constant is given by $k = \frac{\lambda}{l_0}$ where λ is the modulus of elasticity and l_0 is the natural length. On substitution into $T = W$ we obtain

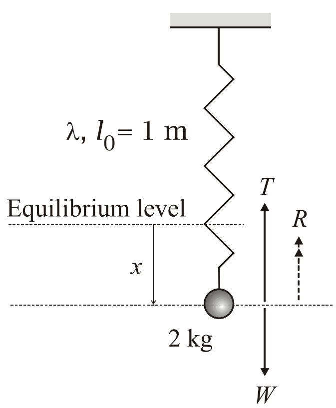
$$\frac{\lambda}{l_0} d = mg$$

Here $d = 0.24$ m, $l_0 = 1.0$ m, $m = 2kg$ and $g = 9.8$ ms^{-2}. Hence

$$\frac{\lambda}{1} \times 0.14 = 2 \times 9.8$$

$$\lambda = \frac{2 \times 9.8}{0.24} = \frac{245}{3} \text{ N}$$

(*b*)



Let x represent the extension of the particle beyond the equilibrium level. The two forces acting on the particle continue to be the weight $(W = mg)$ and the tension $(T = ky)$ where y is the total extension – that is, the extension up to the equilibrium point (0.24 m) followed by the extension x beyond the equilibrium point. Since the particle is extended beyond the equilibrium point it experiences a resultant force pulling it upwards.

$R = W - T$

$R = mg - ky$

But $y = x + 0.24$. Hence

$R = mg - k(x + 0.24)$

$R = mg - kx - 0.24k$

But from the first part of the question $mg = 0.24k$. Hence,

$R = -kx$

The resultant force obeys Newton's second law, hence

$$m\frac{d^2x}{dt^2} = -kx$$

which is the equation of simple harmonic motion. Here $k = \frac{245}{3}$, $m = 2$ giving

$$2\frac{d^2x}{dt^2} = -\frac{245}{3}x$$

$$\frac{d^2x}{dt^2} = -\frac{245}{6}x$$

Since the particle is released from a position 0.3 m below the equilibrium position the amplitude at $t = 0$ is $A = 0.3$ and the solution to the equation $\frac{d^2x}{dt^2} = -\omega^2 x$ is $x = 0.3\cos\omega t$ where A is the amplitude. Hence

$$x = 0.3\cos\left(\sqrt{\frac{245}{6}}t\right) = 0.3\cos\left(7\sqrt{\frac{5}{6}}\,t\right)$$

The particle returns to the equilibrium position when $x = 0$. To find the time when this happens, the period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\sqrt{\frac{245}{6}}\right)}$. The time after which the particle passes through the equilibrium position for the first time is $\frac{1}{4}$ the period: $t_1 = \frac{\pi}{2\left(\sqrt{\frac{245}{6}}\right)} = 0.2458... = 0.256\text{ s}$ (3 s.f.). The maximum speed of the particle when it passes through the equilibrium position is given by

$$v_{\max} = \omega A = \sqrt{\frac{245}{6}} \times 0.3 = 1.9170... = 1.92\text{ ms}^{-1} \quad \text{(3 s.f.)}$$

Energy in simple harmonic motion

Let us take an oscillating mass $(m\,\text{kg})$ attached to a light spring as an example of simple harmonic motion. With x as the extension of the spring beyond the equilibrium point and v as the mass's velocity, this mass will have three kinds of energy.

(1) Elastic potential energy given by

$E = \frac{kx^2}{2}$ $\qquad k = \frac{\lambda}{l_0}$, spring constant $\qquad \lambda$, Young's modulus $\qquad l_0$, natural length

(2) Gravitational potential energy, given by

$U = mgh$ $\qquad h$ is the height of the mass above a reference level

(3) Kinetic energy, given by

$E_k = \frac{1}{2}mv^2$

As the object oscillates, these forms of energy will be inter-converted. Let us assume that this is a closed system, so that total energy within it is conserved and there are no energy losses from it; in particular, we assume that there is no friction to dampen the motion of the mass. Hence

$$\text{total energy} = \begin{pmatrix}\text{gravitational}\\ \text{potential energy}\end{pmatrix} + \begin{pmatrix}\text{kinetic}\\ \text{energy}\end{pmatrix} + \begin{pmatrix}\text{elastic potential}\\ \text{energy}\end{pmatrix}$$

$$E = mgh + \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Suppose we set the reference point for the determination of the gravitational potential energy in $U = mgh$ to be the level when the particle is at its lowest point, then at this point the particle will have 0 gravitational potential energy, 0 kinetic energy and maximum elastic potential energy. As the particle accelerates towards the equilibrium level, it will lose elastic potential energy, which will be converted to gravitational and kinetic energy. When it passes the point where there is no extension or compression of the spring, the elastic potential energy will be 0, and the object will have both gravitational potential energy and kinetic energy. The kinetic energy is converted to elastic potential energy, as the spring is compressed and to gravitational potential energy, as the object rises to its maximum height. These energy conversions are reversed while the object is moving downwards.

Energy conversions can be used to solve certain problems concerning harmonic oscillators.

Example (1) continued

(*c*) With the system as defined in example (1), by use of energy considerations, determine how high above the point of release the particle reaches during the period of harmonic motion.

Solution

(*c*) Let h be the maximum height above the equilibrium point that the particle reaches. The particle's elastic potential energy is

$$\text{Elastic potential energy} = \frac{\lambda x^2}{2l_0}$$

where x is the extension. At the equilibrium point $x = 0.24 \text{ m}$. We saw in part (*b*) of the question that $v_{\max} = 1.9170... \text{ ms}^{-1}$. This is the velocity of the particle has it passes through the equilibrium point. Its kinetic energy at this point will be

$$K_E = \frac{1}{2}mv^2 = \frac{1}{2} \times 2 \times (1.9170...)^2$$

As the particle passes the equilibrium point it gains gravitational potential energy, but loses kinetic energy and elastic potential energy.

Gain of gravitational potential = loss in elastic potential + loss in kinetic energy

$$mgh = \frac{1}{2}mv^2 + \frac{\lambda x^2}{2l_0}$$

Therefore, on substituting $m = 2,\ g = 9.8,\ \lambda = \frac{245}{3}$ and $l_0 = 1$

$$2 \times 9.81 \times h = \frac{1}{2} \times 2 \times (1.9170...)^2 + \frac{\left(\frac{245}{3}\right) \times (0.24)^2}{2}$$

$$19.6h = 6.027$$

$$h = 0.3075 \text{ m}$$

The height above release is $0.3 + 0.3075 = 0.6075 = 0.608 \text{ m}\ (3 \text{ s.f.})$