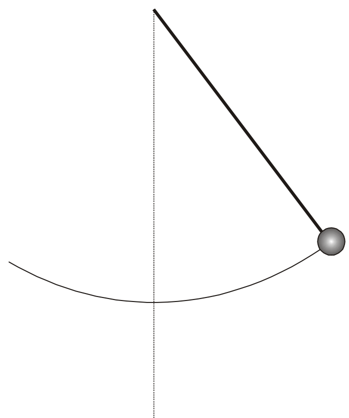
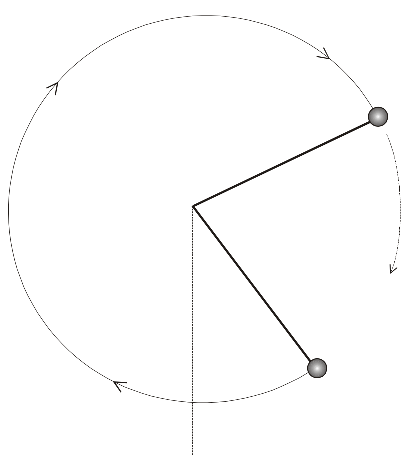
Simple pendulum

A simple pendulum is a bob or object with mass suspended by an inextensible string or rod from a point, where the connection is frictionless.



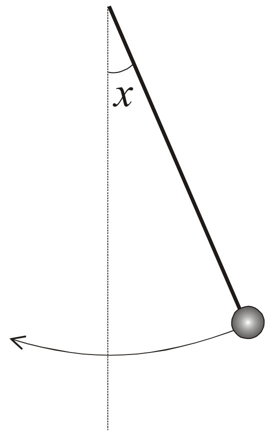
When the bob is displaced from the vertical, it will oscillate.

In fact, these oscillations do not obey simple harmonic motion. One way to see intuitively that the simple pendulum does not in fact oscillate with simple harmonic motion is to realise that if the bob is given sufficient energy it will complete a whole revolution, and carry on indefinitely in one direction – assuming that the connection is frictionless, and the object is in a vacuum.



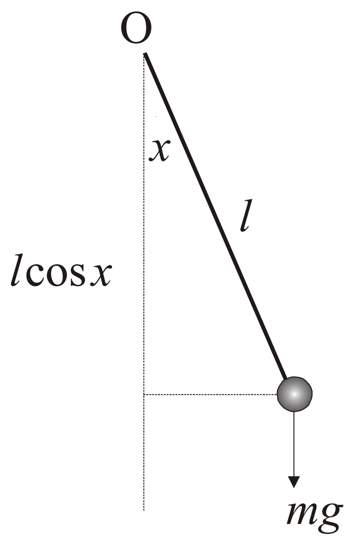
So as the pendulum system acquires more and more energy, its motion becomes less and less like a simple harmonic oscillator.

Nevertheless, if the angle of the swing of the pendulum is small, a pendulum does behave approximately like a simple harmonic oscillator, which we will now show.



That is, we require the angle, x, in the above diagram to be small.

We will use energy considerations in order to derive the pendulum equation.



Let the pendulum rod be attached to a fixed point, O. Let the length of the pendulum be l, and the angle made by the pendulum rod with the vertical axis be x. Then the vertical distance of the bob below the fixed point, O, will be $l\cos x$.

Now gravitational potential energy = weight × height above a reference point

$U = mgh$

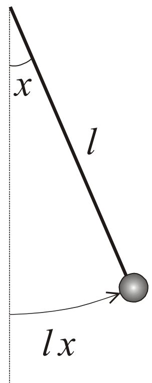
Taking the level of the fixed point, O, to be zero gravitational potential energy, the bob, therefore, has

$$U = -mgl\cos x$$

potential energy at the point where the rod makes the angle, x, with the vertical.

The velocity will be measured in the same direction as the angle, x. The arc length is

$s = lx$



so the velocity is

$v = l\dot{x}$

where $\dot{x} = \frac{dx}{dt}$, the derivative of the angular displacement with respect to time.

Hence, the kinetic energy of the bob is

$$E_K = \tfrac{1}{2}mv^2 = \frac{m(l\dot{x})^2}{2} = \frac{ml^2(\dot{x})^2}{2}$$

As the whole system is frictionless, no energy is lost from it, so conservation of energy implies that total energy is conserved.

$$\begin{pmatrix}\text{total}\\\text{energy}\end{pmatrix} = \begin{pmatrix}\text{kinetic}\\\text{energy}\end{pmatrix} + \begin{pmatrix}\text{gravitational}\\\text{potential energy}\end{pmatrix}$$
$$E = E_K + U$$

Hence,

$$E = \frac{ml^2(\dot{x})^2}{2} - mgl\cos x$$

where E is a constant.

Differentiating both sides with respect to x gives

$$\frac{ml^2}{2} \times 2\dot{x} \times \frac{d\dot{x}}{dx} + mgl\sin x = 0$$

Hence,

$$ml^2 \times \dot{x} \times \frac{d\dot{x}}{dt} + mgl\sin x = 0 \qquad (1)$$

Some notes about this process of differentiation. Firstly, energy, *E*, is a constant, so its derivative is zero. Secondly, we have here differentiated with respect to *x,* the angular displacement, and *not* with respect to *t*, time. Thirdly, you are reminded that the derivative of $\cos x$ is $-\sin x$.

Now $\dot{x} = \frac{dx}{dt}$

So equation (1) becomes

$$ml^2 \times \frac{dx}{dt} \times \frac{d\dot{x}}{dx} + mgl\sin x = 0$$

Cancelling through the *dx* terms

$$ml^2 \times \frac{dx}{dt} \times \frac{d\dot{x}}{dx} + mgl\sin x = 0$$

gives

$$ml^2 \times \frac{d\dot{x}}{dt} + mgl\sin x = 0$$

Hence,

$$ml^2\ddot{x} + mgl\sin x = 0 \qquad \text{PENDULUM EQUATION}$$

This is the pendulum equation.

Dividing through by ml^2 gives

$\ddot{x} + \frac{g}{l}\sin x = 0$

Letting $k = \frac{g}{l}$ then

$\ddot{x} + k\sin x = 0$ SIMPLIFIED PENDULUM EQUATION

which is the simplified pendulum equation.

Since x is small we can approximate x by $\sin x$, hence, on substitution, we obtain

$\ddot{x} + kx = 0$ HARMONIC APPROXIMATION

which is the harmonic approximation to the pendulum equation.

This is the equation of a harmonic oscillator, as rearrangement will quickly show

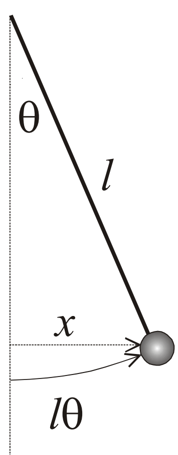
$\ddot{x} = -kx$

That is, acceleration is proportional to displacement and acts in the opposite direction. The harmonic approximation to the pendulum equation, therefore, has solution

$x(t) = a\sin(\omega t + \alpha)$

where a is the amplitude, $\omega = \sqrt{k} = \sqrt{\frac{g}{l}}$ is the angular frequency, and α is the phase angle.

In this equation x represents the angular displacement. But the same equation could equally describe the motion of the pendulum if we used the horizontal displacement of the bob from the vertical axis.



This is because, when the angular displacement, here θ, is small, then the horizontal displacement of the bob from the vertical axis is also approximately given by

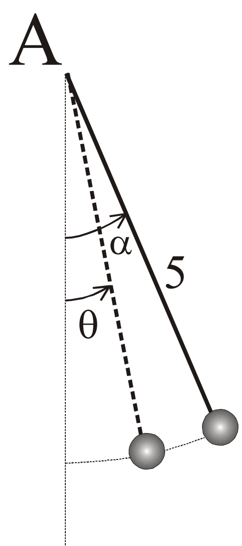
$$x = l\theta$$

which means that the same equation

$$\ddot{x} = -kx$$

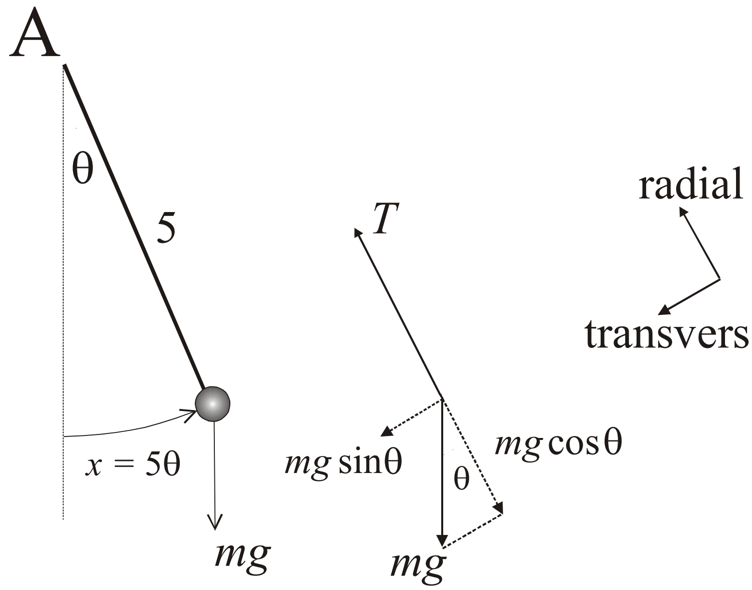
describes the motion of a pendulum whether x measures the angular displacement or the horizontal displacement, provided the displacements are small.

<u>Example</u>



A bob is suspended from a fixed point A by means of a light inextensible string, which has length $5m$. The bob is pulled up so that it makes an angle α with the downward vertical line through A. It is then released from rest. Let θ be the angle made by the string with the downward vertical through A after t seconds. (*i*) Find the transverse component of the acceleration of the particle at time t. (*ii*) Obtain a differential equation involving $\frac{d^2\theta}{dt^2}$. (*iii*) Use the approximation $\sin\theta \approx \theta$ to show that when α is small, and when air-resistance is ignored, the bob performs simple harmonic motion. (*iv*) Find the approximate period of the oscillation, giving your answer to 2 significant figures.

Solution



(i) The transverse component of the weight of the bob is = $mg\sin\theta$.

(ii) Let x denote the displacement of the bob along the arc of its path from the vertical axis. Then

$$x = r\theta = 5\theta$$

The acceleration of the bob in the transverse direction is in the opposite direction to the displacement, and is given by

$$\text{transverse acceleration} = \frac{d^2x}{dt^2}.$$

Since $F = ma$ then

$$-gm\sin\theta = m\frac{d^2x}{dt^2}$$

The negative sign indicates that the force acts in the opposite direction to the displacement. On simplifying, we obtain

$$\frac{d^2x}{dt^2} = -g\sin\theta$$

On substituting $x = 5\theta$ we obtain

$$\frac{\mathrm{d}^2(5\theta)}{dt^2} = -g\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g\sin\theta}{5}$$

which is a differential equation involving $\frac{d^2\theta}{dt^2}$.

(iii) If θ is small, and ignoring air-resistance, we obtain

$$\frac{\mathrm{d}^2\theta}{dt^2} = -\frac{g\theta}{5}$$

which is the equation for simple harmonic motion.

(iv) The solution of this equation is

$$\omega = \sqrt{\frac{g}{l}} = \sqrt{\frac{9.81}{5}} = 1.4 rads^{-1} (2.S.F.)$$

Hence,

$$T = \frac{2\pi}{\theta} = 4.5s (2.S.F.)$$