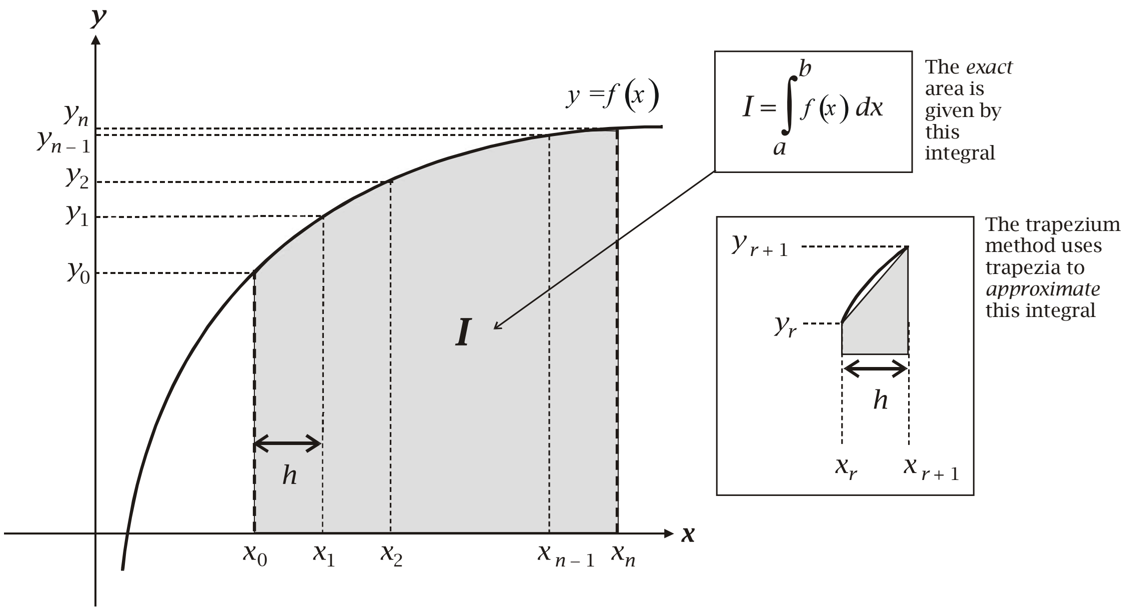
Simpson's Method

Prerequisites

You should be familiar with the trapezium (also called *trapezoidal*) method for finding the area under a curve. The *exact* area under the curve $y = f(x)$ from a to b is denoted by the symbol

$$I = \int_a^b f(x)\,dx$$

It is called the *exact integral* of $y = f(x)$ from a to b. The *trapezium rule* is a general formula for approximating the area under the graph of the function $y = f(x)$ by trapezia of width h between ordinates x_0 and x_n.



The trapezium rule is

$$A = \frac{h}{2}\left\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right\}$$

where the width of the trapezia is given by $h = \dfrac{b-a}{n}$ where a and b are the values of the first and last x ordinates respectively and there are n intervals (giving $n + 1$ ordinates).

Example (1)

Use the trapezium method with an interval width of 0.1 to determine the approximate value of

$$I = \int_2^3 \ln x dx$$

Give your answer to 6 decimal places.

Solution

The trapezium rule is

$$I \approx \int_a^b f(x)dx \approx \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$$

Here $h = 0.1$. There are 11 ordinates.

$$\int_a^b f(x)dx \approx \frac{0.1}{2}\{(\ln(2.0) + \ln(3.0)) + 2(\ln(2.1) + \ln(2.2) + \ldots + \ln(2.9))\}$$

$$= 0.0.5\{(0.6931\ldots + 1.0986\ldots) + 2(0.7419\ldots + 0.7884\ldots + \quad \ldots. \quad + 1.0647\ldots)\}$$

$$= 0.90940364\ldots$$

$$= 0.909404 \text{ (6 d.p.)}$$

Numerical integration

The trapezium method is an example of a technique of *numerical integration.* To integrate is to find the area under the curve of a function $y = f(x)$. The theory of the integral calculus develops techniques for finding the *exact* integral, denoted, as above, by $I = \int_a^b f(x)dx$. The theory of the integral calculus is based on the *Fundamental theorem of calculus*, which states that integration is the reverse process of differentiation. However, (1) it is not always possible to find the exact integral of a function $y = f(x)$; (2) the theory of the integral calculus is based on the theory of approximations to the exact area. In other words, it is essential to study methods of obtaining an approximate solution to the problem of finding the area under a curve. These approximate methods are generally known as *numerical integration.*

The obvious theoretical question with a numerical method is *how good an approximation is it?* Suppose we denote an exact integral by I

$$I = \int_a^b f(x)dx$$

and an approximation to it by A. Then the *absolute error* introduced by approximating I by A is the size (or *modulus*) of the difference

$\text{error} = |I - A|$

This is an indication of the error, but more useful is the *relative error*, or *percentage relative error*, given by

$$\text{percentage relative error} = \frac{\text{absolute error}}{\text{true value}} \times 100\%.$$

This is more useful precisely because it is *relative* to the size of the integral *I*.

Example (1) continued

Find to 5 decimal places the percentage relative error in using the approximation of example (1).

Solution

The approximate value of the integral in example (1) was found to be

$$\int_a^b f(x)\,dx \approx 0.90940364...$$

The exact value of the integral is given by

$$\begin{aligned}\int_2^3 \ln x\,dx &= [x\ln x - x]_2^3 \\ &= (3\ln 3 - 3) - (2\ln 2 - 2) \\ &= 0.909542504...\end{aligned}$$

$$\begin{aligned}\text{Percentage relative error} &= \frac{\text{absolute error}}{\text{true value}} \times 100\% \\ &= \left(\frac{0.909542504... - 0.90940364...}{0.909542504...}\right) \times 100\% \\ &= 0.01523\% \text{ (5 d.p.)}\end{aligned}$$

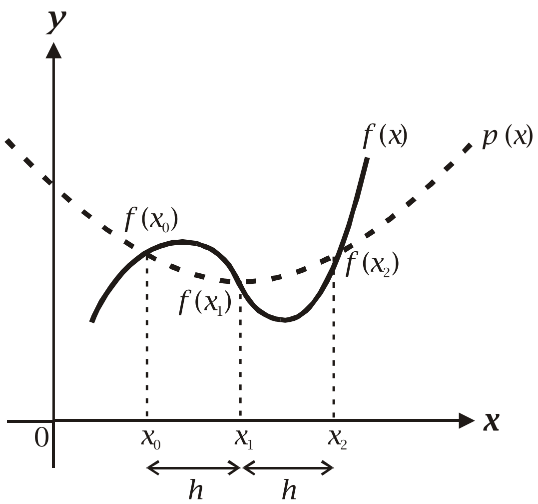
It is an error of not more than 0.1%. That would seem to be quite good, but an approximation is only good if it meets the demands of the practical application. In certain contexts, a relative error around 0.1% would be *not good enough!* Furthermore, whilst computers can be programmed to do these things, so taking away some of the burden of the labour involved, in example (1) we used an interval width of 0.1 and the calculation of the values of the ordinates was quite labour intensive. Even computer *algorithms* (programmes for processing data) can exhaust the processing capacities of the computer. So we want better and better numerical approximations to the exact integral, and we want those approximations to be efficient, in the sense of avoiding unnecessary computation. We can always get a better approximation by decreasing the size of the interval, which requires more and more calculations, but can we get a better approximation *without* taking a smaller interval size? The answer to that question is "yes". There are several improvements on the trapezium method, and one of these is *Simpson's Method*.

Simpson's method

Simpson's method is an extension to the trapezium method that tries to get closer to the real shape of the curve that we are approximating by using a curve rather than a line. In Simpson's method a quadratic function is used to approximate the shape of the function we are trying to integrate. That means, we are trying to fit small segments of parabolas to the shape of the real curve rather than straight lines as we do in the trapezium method. Suppose we have three equally spaced points

$(x_0, y_0) = (x_0, f(x_0))$
$(x_1, y_1) = (x_1, f(x_1))$
$(x_2, y_2) = (x_2, f(x_2))$

on a curve $y = f(x)$. Let the interval between these points be *h.* The curve $y = f(x)$ between (x_0, y_0) and (x_2, y_2) may be twisting and turning in any number of ways – in the diagram below we have shown it change direction twice between these two points. We will approximate the shape of this curve by a parabola passing through all three points – shown in the diagram as $p(x)$. The shape of the parabola is fixed by the three points and is given by a quadratic function $p(x) = ax^2 + bx + c$ where *a, b, c* are real numbers. The approximation is the area under the parabola.



We state that the area of this parabola is given by

$I \approx \frac{h}{3}(f(x_0)) + 4f(x_1) + f(x_2)$.

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(To prove it we would find the use of Gaussian row reduction from vector algebra useful, which is not a prerequisite of this chapter.[1])

Example (2)

Use the formula

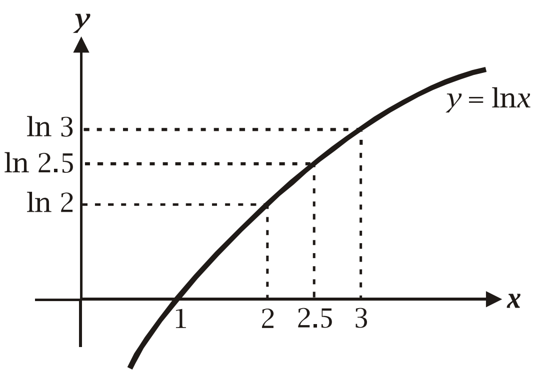
$$I \approx \frac{h}{3}\left(f(x_0)\right) + 4f(x_1) + f(x_2)$$

to find an approximation to

$$I = \int_2^3 \ln x dx$$

with two intervals. Give your answer to 6 decimal places. Given that the real value of the integral is 0.9095425 to 7 decimal places, find to 5 decimal places the percentage relative error created as a result of using this approximation.

Solution



Here $h = \frac{1}{2}$

$$\begin{aligned} I &\approx \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) \\ &= \frac{1}{6}\left(\ln(2) + \ln(2.5) + \ln(3)\right) \\ &= \frac{1}{6}\left(0.69314718... + 4(0.916290731...) + 1.098612289...\right) \\ &= 0.9094871 \quad \text{(7 d.p.)} \end{aligned}$$

[1] There is a separate chapter at blacksacademy.net entitled *Proof of Simpson's Formula.*

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$$\text{Percentage relative error} = \frac{\text{absolute error}}{\text{true value}} \times 100\%$$
$$= \left(\frac{0.9095425 - 0.9094871}{0.9095425}\right) \times 100\%$$
$$= 0.00609\% \text{ (5 d.p.)}$$

Comparing this result with the result obtained previously using the trapezium method.

Trapezium method with 10 intervals	% relative error	0.01523
Simpson's method with 2 intervals	% relative error	0.00609

Simpson's method with only *two* intervals has already improved upon the trapezium method used with *ten* intervals. There has been an increase in both the efficiency of the method and its accuracy. We also expect to increase the accuracy by using Simpson's formula in composite form, which means, using several smaller intervals at the same time. The formula for Simpson's method in composite form approximates the exact integral $I = \int_a^b f(x)\,dx$ by

$$I = \int_a^b f(x)\,dx \approx \frac{h}{3}\{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n)\}$$

where $x_r = a + rh$. The rule may also be written

$$I = \int_a^b f(x)\,dx \approx \frac{h}{3}\{y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 4y_{n-1} + y_n\}$$

where $y_r = f(x_r)$.

The nature of this formula means that the minimum number of intervals that can be used is 2 (3 ordinates). To increase the accuracy of the approximation intervals must be added in multiples of 2.

Example (3)

(*a*) Use Simpson's Method to find an approximation to $I = \int_2^3 \ln x\,dx$ with interval width $h = \frac{1}{10}$. Give your answer to 7 decimal places.

(*b*) Find to 5 decimal places the percentage relative error value given that the exact value is 0.9095425 to 7 decimal places

Solution

(*a*) $$I \approx \frac{h}{3}\{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n)$$
$$= \frac{0.1}{3}\left\{\begin{aligned}&\ln(2) + 4\ln(2.1) + 2\ln(2.2) + 4\ln(2.3) + 2\ln(2.4) + 4\ln(2.5)\\ &\qquad +2\ln(2.6) + 4\ln(2.7) + 2\ln(2.8) + 4\ln(2.9) + \ln(3.0)\end{aligned}\right\}$$
$$= 0.909542407\ldots$$
$$= 0.9095424 \text{ (7 d.p.)}$$

(b) Relative error $= \dfrac{0.9095425 - 0.9095424}{0.9095425}$
$= 0.00001\% \quad (5 \text{ d.p.})$

The improved accuracy and efficiency of this method over the trapezium method is illustrated as follows.

Trapezium method with 10 intervals	% relative error	0.01523
Simpson's method with 2 intervals	% relative error	0.00609
Simpson's method with 10 intervals	% relative error	0.00001

This example has illustrated the efficacy of Simpson's method. However, under exam conditions it is rare to be asked to find an approximation with as many as ten intervals. The following example is more typical of those met in examinations.

Example (4)

Use Simpson's Rule with five ordinates to find an approximate value for

$$I = \int_1^2 \sqrt{3 + x^3}\, dx$$

Show your working and give your answer to three decimal places.

Solution

Here the interval width is $h = 0.2$. Simpson's rule is

$$I = \int_a^b f(x)dx \approx \frac{h}{3}\{y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 4y_{n-1} + y_n\}$$

Computing the ordinates to 5 decimal places

$$y = \sqrt{3 + x^3}$$
$$y_0 = \sqrt{3 + 1^3} = 2$$
$$y_1 = \sqrt{3 + (1.25)^3} = 2.22556$$
$$y_2 = \sqrt{3 + (1.5)^3} = 2.52488$$
$$y_3 = \sqrt{3 + (1.75)^3} = 2.89126$$
$$y_4 = \sqrt{3 + (2)^3} = 3.31662$$

Substituting into Simpson's rule

$$I \approx \frac{0.25}{3}\{2 + (4 \times 2.22556) + (2 \times 2.52488) + (4 \times 2.89126) + 3.31662\}$$
$$= 2.56947\ldots$$
$$= 2.570 \ (3\text{d.p})$$

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