Simulation by Monte Carlo Methods

The need for simulations

An Academy proposes to situate a new school in a rural area. They submit a planning application to the local authority. The local residents mainly object on the grounds that the new school will cause traffic congestion in the area. One junction in particular is in question. The directors of the Academy wish to establish that the modest increase of traffic at the junction will cause no congestion.

The only way to do this is by simulation. A simulation attempts to create a picture of what will happen at the junction under the conditions of an increase in the number of cars.

Simulations depend on some process of simulating the effect of chance on random events. Since casinos have similar methods of simulating chance – a roulette table for instance – and since Monte Carlo is the most famous of all (European) gambling resorts – the process of making a simulation is often called a Monte Carlo method. Also, simulations depend on chance and any process that depends on chance is called a *stochastic* process.

Random Sampling

Simulations work by a means of random sampling. For example, suppose that you know that on average ten cars arrive at a junction along a given road in any given period of time. Assuming that the cars arrive independently of each other – that is, the arrival of one car does not make the arrival of another more likely – this means the probability of a car arriving at any given second is $\frac{1}{6}$ or 10 out of 60.

In order to run the simulation of cars arriving at the junction we could throw a die to represent each second. If the number comes up a 1 then we will say that a car has arrived at that second; otherwise, there will be no car.

Example

Simulate the arrival of cars at the junction along the given road over a 60 second period by using a six sided as suggested above.

Solution

This requires you to throw a die 60 times. The successive throws of the die represent the successive seconds of the period of the simultation. I did this and obtained the following results

Throw = second	Value	Arrival?
1	1	yes
2	1	yes
3	6	
4	1	yes
5	6	
6	6	
7	3	
8	1	yes
9	5	
10	3	
11	5	
12	5	
13	2	
14	1	yes
15	4	
16	3	
17	1	yes
18	5	
19	5	
20	3	
21	4	
22	5	
23	6	
24	3	
25	4	
26	3	
27	2	
28	5	
29	3	
30	1	yes

Throw = second	Value	Arrival?
31	3	
32	1	yes
33	2	
34	2	
35	3	
36	6	
37	3	
38	4	
39	5	
40	5	
41	5	
42	1	yes
43	2	
44	2	
45	3	
46	4	
47	4	
48	2	
49	4	
50	5	
51	3	
52	5	
53	1	yes
54	2	
55	3	
56	2	
57	2	
58	2	
59	5	
60	1	yes

In this simulation cars are expected to arrive in the following seconds

1, 2, 4, 8, 14, 17, 30, 42, 53, 60

Random Number Sampling

Another approach is to use a table of random numbers. Each element of the population is assigned a number. Tables of random numbers are used to select the sample. The table is constructed so that each digit is equally likely to appear. You ignore numbers that fall outside the specified range.

Example

In a simulation of traffic arriving at a road junction, it is known that there are on average ten cars arriving at the junction along one road per minute. The cars arrive, however, at random. Divide the minute into sixty seconds and use the table of random numbers below to determine in which of these sixty seconds a car will arrive in the simulation.

93	89	09	57	07	14
93	40	81	06	85	85
16	01	19	69	11	78
26	52	89	13	86	00
53	32	90	43	79	01
07	35	73	60	55	82
77	89	52	48	33	72
01	62	76	42	71	92
12	30	97	86	96	43
60	37	34	69	41	69
95	45	90	32	78	52
05	42	41	08	34	67
09	29	69	55	39	90
22	00	62	97	03	18
62	31	82	15	73	90
58	78	45	08	90	33
95	15	67	49	54	81

Solution

The table gives a range of 100 random numbers from 00 to 99. We require only 60 such numbers, so any number in the range 60 to 99 will simply be ignored. Of these 60 the probability of a car arriving is $\frac{1}{6}$, or 10 out of 60. We will say a car will arrive if the value of the random number lies between 00 and 09; otherwise the random number will represent a second in which there was no car arriving.

The numbers in the table that we will not use are greyed out.

93	89	39	57	07	14
93	40	81	26	85	85
16	01	19	69	11	78
26	52	89	13	86	00
53	32	90	43	79	01
07	35	73	59	55	82
77	89	52	48	33	72
01	62	76	42	71	92
12	30	97	86	96	43
60	37	34	69	41	69

95	45	90	32	78	52
45	42	41	08	34	67
09	29	69	55	39	90
22	00	62	97	03	18
62	31	82	15	73	90
58	78	45	08	90	33
95	15	67	49	54	21

Second	Random No.	Arrives
1	39	No
2	57	No
3	07	Yes
4	14	No
5	40	No
6	26	No
7	16	No
8	01	Yes
9	19	No
10	11	No
11	26	No
12	52	No
13	13	No
14	00	Yes
15	53	No
16	32	No
17	43	No
18	01	Yes
19	07	Yes
20	35	No
21	59	No
22	55	No
23	52	No
24	48	No
25	33	No
26	01	Yes
27	42	No
28	12	No
29	30	No
30	43	No

Second	Random No.	Arrives
31	37	No
32	34	No
33	69	No
34	41	No
35	45	No
36	32	No
37	52	No
38	45	No
39	42	No
40	41	No
41	08	Yes
42	34	No
43	09	Yes
44	29	No
45	55	No
46	39	No
47	22	No
48	00	Yes
49	03	Yes
50	18	No
51	31	No
52	15	No
53	58	No
54	45	No
55	08	Yes
56	33	No
57	15	No
58	49	No
59	54	No
60	21	No

Cars are simulated to arrive at 3, 8, 14, 18, 19, 26, 41, 43,48, 49 and 55 seconds.

Note that instead of using tables of random numbers you could use a random number generator – on your calculator for instance. On my calculator pressing Shift followed by a dot produces a three digit random number. The first two digits can be used for a two digit random number and the remaining digit either discarded or used for the first digit of the next two digit random number. Of course, you may only require one digit, or you may require a three digit number depending on what you have to simulate.

The advantages of random number sampling are (a) that it is random and free from bias and (b) every number has equal chance of selection

The disadvantage is that it is not suitable for large samples.

Simulating queues – the queuing discipline

Simulations are often used to predict what will happen in a queuing situation. But in order to complete the simulation you require a set of rules called the *queuing discipline* governing what happens in the queue. Two familiar rules that might be applied are

FIFO First in, first out

LIFO Last in, first out

The *FIFO* situation involves the idea of a priority queue. People join at the back of the queue and are served from the front.

In the *LIFO* situation the queue is like a stack of plates. The plates can only be removed from the top, so the first plate to be removed is the last one to be placed on the stack.

Example

In the situation above it takes on average 4 seconds for a car to leave the junction. The probability of leaving the junction as a function of the number of seconds the car has been at the junction is given in the following table.

waiting time / s	1	2	3	4	5	6	7
probability	0.1	0.1	0.2	0.2	0.2	0.1	0.1

Using the simulation of the cars arriving at the junction in the above example and simulating the waiting time at the junction using a table of random numbers from 0 to 9 (or otherwise) estimate during how many seconds in a minute there are (a) two or more cars waiting at the junction (b) three or more cars waiting at the junction

Solution

We will use a choice 10 random digits (0 to 9) to simulate the waiting times, according to the following table

time / s	1	2	3	4	5	6	7
probability	0.1	0.1	0.2	0.2	0.2	0.1	0.1
digit	0	1	2,3	4,5	6,7	8	9

This means that if a digit 3 is drawn, for example, the simulated waiting time will be 3 seconds.

Using this table, we arrived at the following simulation of the waiting times for the cars that are expected to arrive at the junction at 3, 8, 14, 18, 19, 26, 41, 43, 48, 49 and 55 seconds.

Car	Second of arrival	Random digit	Waiting time
1	3	0	1
2	8	4	4
3	14	2	2
4	18	4	4
5	19	4	4
6	26	9	7
7	41	7	5
8	43	7	5
9	48	3	3
10	49	1	2
11	55	1	2

Then the simulation proceeds as in the following table, which should be relatively self-explanatory. The boxes where a car arrives are greyed in. The light grey box represents the top car waiting at the junction. As cars build up in the queue at the junction, they are represented by the darker shading.

Second	1	2	3	4	5	6	7	8	9	10
Car			1					2	2	2
Second	11	12	13	14	15	16	17	18	19	20
Car	2			3	3			4	4	4
									5	5
Second	21	22	23	24	25	26	27	28	29	30
Car	4	5	5	5	5	6	6	6	6	6
	5									

Second	31	32	33	34	35	36	37	38	39	40
Car	6	6	6							
Second	41	42	43	44	45	46	47	48	49	50
Car	7	7	7	7	7	8	8	8	8	8

			8	8	8			9	9	9
									10	10
Second	51	52	53	54	55	56	57	58	58	60
Car	9	9	10	10	11	11				
	10	10								

The result of the simulation is that there are (a) two or more cars waiting at the following seconds: 19, 20 21, 43, 44, 45, 48, 49, 50, 51, 52 – 11seconds in the given simulated minute of activity. (b) There are three or more cars waiting at the junction during the 49^{th} and 50^{th} seconds – 2 seconds in the given simulated minute of activity.

Running the simulation more than once

Strictly speaking a simulation should be run several times in order to establish the stability and reliability of the solution. There may be freak occurrences, such as the arrival at the junction of five cars in successive seconds, all with long waiting times, and we would want the simulation to thrown up such situations and show us how likely they are to lead to problematic queues and so forth. Running the simulation many times could be very time consuming, so we might want to look for labour saving approaches.

The most time consuming aspect is writing out, for example, all the seconds in sixty seconds, when all you really want to know on which seconds a car is simulated to arrive at the junction. One approach to short-cutting the work is to devise or obtain a table of inter-arrival times.

For example, we might know the mean and standard deviation of the inter-arrival times. The mean may be 6 seconds between arrivals (an average of 10 a minute) and the standard deviation 2 seconds. This would give approximately the following probabilities for the inter-arrival times

inter-arrival time / s	2	3	4	5	6	7	8	9	10
probability	0.03	0.07	0.12	0.18	0.20	0.18	0.12	0.07	0.03

We could simulate the arrival times using a set of 100 random digits (00 to 99) as follows

inter-arrival time / s	2	3	4	5	6	7	8	9	10
probability	0.03	0.07	0.12	0.18	0.20	0.18	0.12	0.07	0.03
Random digits from	00	03	10	22	40	60	78	90	97
Random digits to	02	09	21	39	59	77	89	96	99

Using these table I obtained the following results for a simulation

Random digit	Inter-arrival time	Arrival time
27	5	5
05	3	8
36	5	13
72	7	20
88	8	28
58	6	34
12	4	38
60	7	45
45	6	51
51	6	57