Simultaneous differential equations

Systems of differential equations

When dealing with systems of differential equations it is appropriate to use the ‘dot’ notion. That is

$$\ddot{x} = \frac{d^2x}{dt^2}$$
$$\dot{x} = \frac{dx}{dt}$$

and so forth.

An example of a system of differential equations that links two functions $x = x(t)$ and $y = y(t)$ is

$$\ddot{x} = -4x + 3y$$
$$\ddot{y} = 6x - 2y$$

A solution to the first equation necessarily requires a solution to second. In other words we cannot obtain an explicit function $x(t)$ without also simultaneously obtaining an explicit function $y(t)$. We are hee concerned with techniques for solving simultaneous differential equations.

Matrix notation

In order to solve simultaneous differential equations we will use matrix notation.

Therefore, it is important to recall that the system of simultaneous equations in three unknowns x_1, x_2, x_3 as follows

$$2x_1 - 3x_2 + 4x_3 = 6$$
$$x_1 - 6x_2 - x_3 = 9$$
$$-x_1 + 2x_2 + +x_3 = -1$$

can be represented in matrix form by

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & -6 & -1 \\ -1 & 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ -1 \end{bmatrix}$$

If $A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -6 & -1 \\ -1 & 2 & 7 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 6 \\ 9 \\ -1 \end{bmatrix}$

then our simultaneous equations can be written

$$A\mathbf{x} = \mathbf{c}$$

We will use this style of representation for simultaneous differential equations too. That is for example, the simultaneous differential equations

$$3\dot{x} + 4\dot{y} = x - 6y + e^t$$
$$\dot{x} - \dot{y} = 4x + y + \cos t$$

may be written

$$\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ \cos t \end{bmatrix}$$

and if $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$, $B = \begin{bmatrix} 1 & -6 \\ 4 & 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} e^t \\ \cos t \end{bmatrix}$

then we can write this system as

$$A\dot{\mathbf{x}} = B\mathbf{x} + \mathbf{c}$$

In the above example the matrices *A*, *B* are constant coefficient, but in the following example they are functions of t

Example

Write in matrix form

$$t\dot{x}_1 - 2x_3 = \cos t$$
$$\dot{x}_2 - \dot{x}_3 + t^2 x_1 = 0$$
$$(\cos t)\dot{x}_1 - \dot{x}_2 = 2x_3$$

Solution

Rearrange the last equation to $(\cos t)\dot{x}_1 - \dot{x}_2 - 2x_3 = 0$ then

$$\begin{bmatrix} t & 0 & 0 \\ 0 & 1 & -1 \\ \cos t & -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -2 \\ t^2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos t \\ 0 \\ 0 \end{bmatrix}$$

That is $A\dot{\mathbf{x}} + B\mathbf{x} = \mathbf{c}$ where $A = \begin{bmatrix} t & 0 & 0 \\ 0 & 1 & -1 \\ \cos t & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & -2 \\ t^2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} \cos t \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

First order systems

A system is said to be first order if it contains no derivatives higher than the first order. It is linear if there are no combinations such as $x(t)x_2(t)$ - so the functions of the dependant variables are added together and products of the dependant variables are not formed. If the entries in the matrices are constants then the system will be called constant coefficient. A linear constant-coefficient first-order system is said to be in normal form if it is written

$$\dot{\mathbf{x}} = Q\mathbf{x} + \underline{h}(t)$$

If the matrix $\underline{h}(t) = 0$ then the system is said to be homogeneous.

Every constant-coefficient linear first-order system of differential equations can be written in normal form, provided one condition is met. In this case, the general constant-coefficient linear first order system takes the form $A\dot{\mathbf{x}} + B\mathbf{x} = \mathbf{c}(t)$. So, if A is non-singular, that is, comprising of rows (or columns) of linear independent vectors- then A has to be capable of being put into normal form. By multiplying all entries by A^{-1}:

$$A^{-1}A\dot{\mathbf{x}} + A^{-1}B\mathbf{x} = A^{-1}\mathbf{c}(t)$$

Since $A^{-1}A = I$, we get:

$$\dot{\mathbf{x}} + A^{-1}B\mathbf{x} = A^{-1}\mathbf{c}(t)$$

Rearranging

$$\dot{\mathbf{x}} = -A^{-1}B\mathbf{x} + A^{-1}\mathbf{c}(t)$$

Then setting $Q = -A^{-1}\overline{B}$ and $\underline{h}(t) = A^{-1}\mathbf{c}(t)$ we obtain $\dot{\mathbf{x}} = Q\mathbf{x} + \underline{h}(t)$ which is in normal form.

Example

One of these two systems can be place into normal form. Determine which of these two systems can be placed into normal form; explain why the other cannot and find the normal form of the one that can have a normal form.

(i)

$$\begin{aligned} &\dot{x}_1 - \dot{x}_2 + 2\dot{x}_3 + 4x_1 = 3t \\ &-2\dot{x} + 2\dot{x} - 4x_3 + x_2 + x_3 = 6\cos t \\ &\dot{x}_2 + \dot{x}_3 = 1 \end{aligned}$$

(ii)
$$\dot{x}_1 - \dot{x}_2 + 2\dot{x}_3 + 4x_1 = 3t$$
$$-\dot{x}_1 + 2\dot{x}_2 - 4\dot{x}_3 + x_2 + x_3 = 6\cos t$$
$$\dot{x}_2 + \dot{x}_3 = 1$$

<u>Solution</u>

For system (i) the matrix form is

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t \\ 6\cos t \\ 1 \end{bmatrix}$$

We will see immediately in the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \\ 0 & 1 & 1 \end{bmatrix}$ that the second row $(-2,2,-4)$ is a multiple of the first row $(1,-1,-2)$ so A is non-singular and does not have an inverse. For system (ii) the matrix form is

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & -2 & -4 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t \\ 6\cos t \\ 1 \end{bmatrix}$$

We need to find the inverse of $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. We will find this inverse by the formula:

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & \begin{vmatrix} h & i \\ b & c \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ \begin{vmatrix} f & d \\ c & g \end{vmatrix} & \begin{vmatrix} c & g \\ c & a \end{vmatrix} & \begin{vmatrix} c & a \\ f & d \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & \begin{vmatrix} g & h \\ a & b \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$.

Here $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -4 \\ 0 & 1 & 1 \end{bmatrix}$

$$\det A = 1\begin{vmatrix} 2 & 4- \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} -4 & -1 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 6 - 1 - 2 = 3$$

which we can check

$$AA^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -4 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 6 & 3 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

as expected. Our system is

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -4 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t \\ 6\cos t \\ 1 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -4 \\ 0 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{c}(t) = \begin{bmatrix} 3t \\ 6\cos t \\ 1 \end{bmatrix}$

Then pre-multiplying all the matrices by A^{-1}:

$$A^{-1}B = \begin{bmatrix} 6 & 3 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 24 & 3 & 3 \\ 4 & 1 & 1 \\ -4 & -1 & -1 \end{bmatrix}$$

and $A^{-1}\mathbf{c}(t) = \begin{bmatrix} 6 & 3 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3t \\ 6\cos t \\ 1 \end{bmatrix} = \begin{bmatrix} 18t + 18\cos t \\ 3t + 6\cos t + 2 \\ -3t - 6\cos t + 1 \end{bmatrix}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 2t & 3 & 3 \\ 4 & 1 & 1 \\ -4 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18t + 18\cos t \\ 3t + 6\cos t + 2 \\ -3t - 6\cos t + 1 \end{bmatrix}$$

Hence, in normal form the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -24 & -3 & -3 \\ -4 & -1 & -1 \\ 4 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 18t + 18\cos t \\ 3t + 6\cos t + 2 \\ -3t - 6\cos t + 1 \end{bmatrix}$$

Solution to first-order constant coefficient homogeneous systems

A first order constant coefficient system of differential equations has the form $\dot{\mathbf{x}} = Q\mathbf{x} + \mathbf{h}(t)$. If the system is also homogeneous then $\underline{h}(t) = \mathbf{0}$; hence the form is $\dot{\mathbf{x}} = Q\mathbf{x}$. The solution to the system is given by the following theorem. If Q has n linearly independent eigenvectors, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots \mathbf{a}_n$ corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ (where the eigenvalues need not be distinct) then the solution to $\dot{\mathbf{x}} = Q\mathbf{x}$ is $\mathbf{x} = k_1\mathbf{a}_1e^{\lambda_1 t} + k_2\mathbf{a}_2e^{\lambda_2 t} + \ldots + k_n\mathbf{a}_ne^{\lambda_n t}$ where $k_1, k_2 \ldots k_n$ are constants. Before we prove this theorem we will illustrate its application.

Firstly, note that the theorem requires us to begin by placing the system in the form $\dot{\mathbf{x}} = Q\mathbf{x}$. We then proceed by finding the eigenvectors and eigenvalues of the matrix Q. We conclude by substituting into the equation $\mathbf{x} = k_1\mathbf{a}_1e^{\lambda_1 t} + k_2\mathbf{a}_2e^{\lambda_2 t} + \ldots + k_n\mathbf{a}_ne^{\lambda_n t}$.

Example

Solve the system

$$\dot{x}_1 + \dot{x}_2 - 3x_1 - 7x_2 = 0$$
$$-2\underline{x}_1 - x_2 + 4x_1 + 11x_2 = 0$$

<u>Solution</u>

The matrix form is

$$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} -3 & -7 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here let $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

then $A^{-1} = \frac{1}{\det A}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$.

Then $A^{-1}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -3 & -7 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -2 & -3 \end{bmatrix}$

So the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} -1 & -4 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which in normal form is $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ or $\dot{\mathbf{x}} = Q\mathbf{x}$ where $Q = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. We now proceed to find the eigenvalues and eigenvectors of Q. The eigenvalues satisfy the equation. $\det(Q - \lambda I) = 0$. That is

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$
$$(1-\lambda)(3-\lambda) - 8 = 0$$
$$3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$$
$$\lambda^2 - 4\lambda - 5 = 0$$
$$(\lambda - 5)(\lambda + 1) = 0$$
$$\therefore \lambda_1 = 5 \text{ and } \lambda_2 = -1$$

For λ_1 let the eigenvector be $\mathbf{a}_1 = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ then

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$x_1 + 4x_2 = 5x_1$$
$$2x_1 + 3x_2 = 5x_2$$

$-4x_1 + 4x_2 = 0$
$2x_1 - 2x_2 = 0$
$x_1 - x_2 = 0$
$x_1 = x_2$

Therefore $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For λ_2 let the eigenvector be $\mathbf{a}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_1 + 4x_2 = -x_1$
$2x_1 + 3x_2 = -x_2$
$2x_1 + 4x_2 = 0$
$2x_1 + 4x_2 = 0$
$x_1 + 2x_2 = 0 \therefore x_1 = 2x_2$

$\therefore \mathbf{a}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

In summary for $Q = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$\lambda_1 = 5 \quad \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1 \quad \mathbf{a}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Substituting into $\underline{x} = k_1\mathbf{a}_1 e^{\lambda_1 t} + k_2\mathbf{a}_2 e^{\lambda_2 t}$ we obtain $\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = k_1\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}$ where k_1, k_2 are constants. Uncoupling this gives $x_1 = k_1 e^{5t} - 2k_2 e^{-t}$ and $x_2 = k_1 e^{5t} + k_2 e^{-t}$. We will check the solution by substituting back into the original equation which was

$\dot{x}_1 + \dot{x}_2 - 3x_1 - 7x_2 = 0 \qquad (1)$
$-2\underline{x}_1 - x_2 + 4x_1 + 11x_2 = 0 \qquad (2)$

since

$x_1 = k_1 e^{5t} - 2k_2 e^{-t}$
$\dot{x}_1 = 5k_1 e^{5t} + 2k_2 e^{-t}$

and since

$x_2 = k_1 e^{5t} + k_2 e^{-t}$
$\dot{x} = 5k_1 e^{5t} - k_2 e^{-t}$

Substituting into the left hand side of (1):

$$\text{LHS} = \left(5k_1e^{5t} + 2k_2e^{-t}\right) + \left(5k_1e^{5t} - k_2e^{-5}\right) - 3\left(k_1e^{5t} - 2k_2e^{-5}\right) - 7\left(k_1e^{5t} + k_2e^{-t}\right)$$
$$= k_1e^{5t}\left(5+5+3-7\right) + k_2e^{-t}\left(2-1+6+7\right) = 0 = \text{RHS}$$

Substituting into the LHS of (2)

$$-2\left(5k_1e^{5t} + 2k_2e^{-t}\right) - \left(5k_1e^{5t} - k_2e^{-t}\right) + 4\left(k_1e^{5t} - 2k_2e^{-t}\right) + 11\left(k_1e^{5t} + k_2e^{-t}\right)$$
$$= k_1e^{5t}\left(-10-5+4+11\right) + k_2e^{-t}\left(-4+1-8+11\right) = 0 = \text{RHS}$$

So the solution is correct.

The theorem will also apply if their are repeated eigenvalues, provided there are sufficient distinct eigenvectors. That is, so long as the matrix Q in $\dot{\mathbf{x}} = Q\mathbf{x}$ is non-singular.