Simultaneous Equations

Pairs of simultaneous equations

Consider the following equations

$x + 4y = 11$
$2x + y = 8$

Both of the equations contain two unknowns, x and y. The solution to this pair of equations is the values of x and y which satisfy both equations at the same time. The two equations are examples of simultaneous equations. The equation $x + 4y = 11$ is an example of a *linear equation* because in it the variables, x and y, are raised to the power 1 only, and they are not multiplied together. In this context the term *simultaneous equations* denotes a pair of linear equations both in the same two unknown quantities.

Method of elimination

As its name suggests, the aim of this method is to manipulate the equations in such a way that one of the variables is eliminated. This should leave us with one equation containing one unknown, which can be solved easily. Returning to the two equations above.

$x + 4y = 11 \qquad (1)$
$2x + y = 8 \qquad (2)$

If we multiply both sides of the first equation by 2, we get

$2x + 8y = 22 \qquad (3) = (1) \times 2$
$2x + y = 8 \qquad (2)$

$2x$ now appears in both equations. This means that we can eliminate x by subtracting one equation from the other. In the equation $2x + 8y = 22$ we will call the "Left-hand side" of the equation the part to the left of the equals sign $(2x + 8y)$, abbreviated to "LHS", and we will call the part to the right of the equals side (22) the "Right-hand side" of the equation, abbreviated to "RHS". To eliminate x from the simultaneous pair we subtract the left-hand side (LHS) of the second equation from the LHS of the first, and the right-hand side (RHS) of the second equation from the RHS of the first.

$(2x + 8y) - (2x + y) = 22 - 8$
$7y = 14$
$y = 2$

We now have a value for y that simultaneously satisfies both equations. To find the corresponding value of x that satisfies the two equations, we substitute this value of y into either of the original equations; for example, the first.

$$x + 4y = 11$$
$$x + (4 \times 2) = 11$$
$$x + 8 = 11$$
$$x = 3$$

We now have values for both unknowns, and the equations are solved. We can also check the answer. In the final stage of the solution we found x by substituting the value $y = 2$ into the first equation. In this way we found that $x = 3$. However, this pair of solutions should satisfy *both* equations. So that means the values $x = 3,\ y = 2$ should also work in the second equation.

Checking

It is a very good idea to get into the habit of checking a solution wherever possible. In this case the second equation is

$$2x + y = 8 \qquad (2)$$

When $x = 3,\ y = 2$ is substituted into the equation, we should obtain

LHS = RHS

Let us show this. Thus, on substituting $x = 3,\ y = 2$ into the LHS of (2) we get

$$\text{LHS} = (2 \times 3) + 2 = 8 = \text{RHS}$$

So we know we have got this right!

Multiples of both equations

In this problem we needed to carry out only one step before the elimination could occur. Sometimes more than one step is required, as in the following example.

$$2x + 5y = 25 \qquad (1)$$
$$3x - 7y = -6 \qquad (2)$$

This time let us multiply both sides of the first equation by 3 and both sides of the second equation by 2.

$$6x + 15y = 75$$
$$6x - 14y = -12$$

It can be seen that these steps have resulted in the term $6x$ appearing in both equations. We can therefore carry out elimination, again by subtracting one equation from the other.

$$(6x+15y)-(6x-14y)=75-(-12)$$

Removing the brackets and rearranging

$$\cancel{6x-6x}+15y-(-14y)=87$$
$$29y=87$$
$$y=3$$

Now it is relatively simple to substitute this value for y into one of the original equations to find x.

$$2x+5y=25 \qquad (1)$$
$$2x+(5\times 3)=25$$
$$2x+15=25$$
$$2x=10$$
$$x=5$$

Check in (2)

$$3x-7y=-6 \qquad (2)$$

When $x=5,\ y=3$

$$\text{LHS}=(3\times 5)-(7\times 3)=-6=\text{RHS}$$

So we know the solution is correct.

Elimination by adding equations

The elimination is not always carried out using a subtraction. Consider the following example.

$$4x-y=1 \qquad (1)$$
$$3x+2y=20 \qquad (2)$$

Multiplying both sides of the first equation by 2.

$$8x-2y=2 \qquad (1)\times 2$$
$$3x+2y=20 \qquad (2)$$

Now we have $2y$ in one equation and $-2y$ in the other. If we *add* the two equations together, the y terms will cancel each other out.

$$(8x-2y)+(3x+2y)=2+20$$
$$11x=22$$
$$x=2$$

Now we substitute x back into the original first equation to find y.

$(8 \times 2) - 2y = 2$
$16 - 2y = 2$
$16 - 2 = 2y$
$14 = 2y$
$y = 7$

Method of substitution

The aim of this method is to use one of the equations to express one variable in terms of the other. This expression can then be substituted into the other equation, yielding an equation containing only one variable, which can be solved. Consider the following pair of equations

$x - y = 1$ (1)
$2x + 5y = 23$ (2)

This could be solved by multiplying the first equation by 2 and using the method of elimination. However, if we rearrange the first equation, we get

$x = 1 + y$

We have expressed x in terms of y, and can now substitute this expression for x into the second equation.

$2x + 5y = 23$
$2(1 + y) + 5y = 23$
$2 + 2y + 5y = 23$
$7y = 21$
$y = 3$

We can find x conveniently by substituting this value for y into our expression for x:

$x = 1 + y$
$x = 1 + 3$
$x = 4$

This is the method of substitution.

Graphical Method

Consider the following simultaneous equations:

$2y + 6x = 12$ (1)
$y - 4x = -1$ (2)

These can be rearranged to the form $y = mx + c$, the equation of a straight line.

$y = -3x + 6 \qquad (1)'$

$y = 4x - 1 \qquad (2)'$

It would be possible now to replace the y in the first equation with the expression for y in the second equation.

$4x - 1 = -3x + 6$

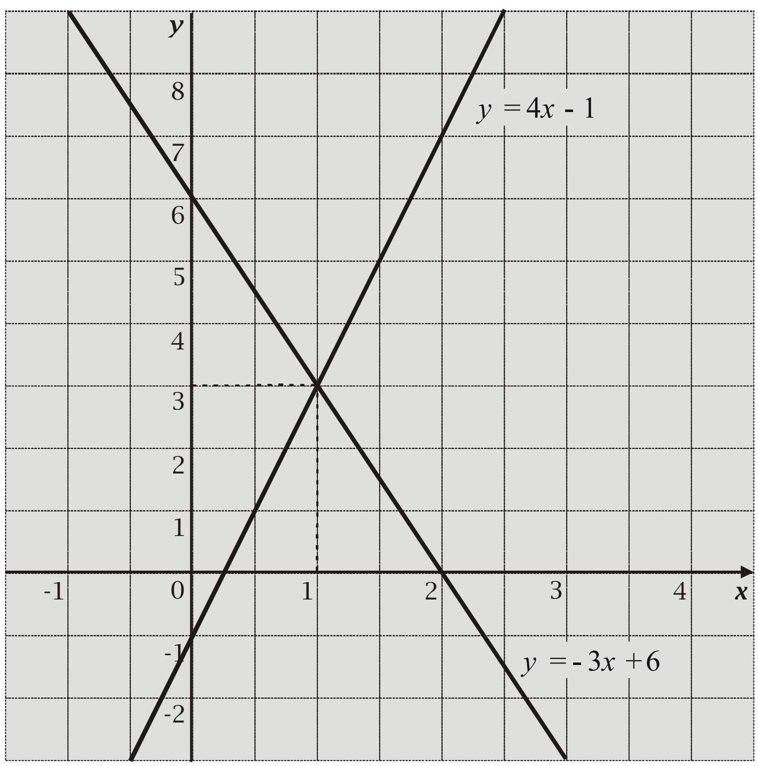
We now have one equation with one unknown, which can be solved.

$4x - 1 = -3x + 6$
$7x = 7$
$x = 1$
$y = 3$

However, the rearrangements

$y = -3x + 6$
$y = 4x - 1$

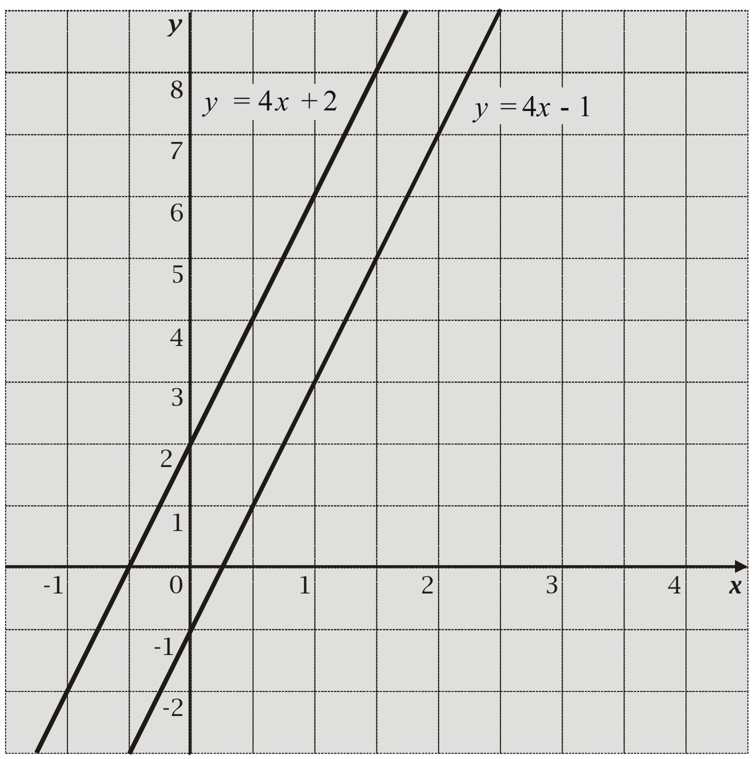
enable us easily to sketch the graphs of the two lines that they represent.



The coordinates of the point of intersection of the two lines give us the values of x and y that simultaneously satisfy the two equations, i.e. $x = 1$ and $y = 3$. The coordinates of every point on each line represent a pair of values of x and y that satisfy the equation of that line. We are looking for a pair of values that satisfy both equations. The point corresponding to this pair of values must therefore lie on both lines. The only place that this occurs is where the lines intersect.

The graphical method is good at explaining why the simultaneous solution to two equations *is* the solution – it corresponds to the single point of intersection of two lines. However, the graphical method is only as accurate as the graph paper you draw on. Consequently, it is best to find the *exact* simultaneous solution of two equations by means of the method of elimination or the method of substitution.

The graphical method does illustrate another important point. Not all simultaneous equations have a solution. For example, when two lines are parallel they do not intersect.



These two lines are parallel. They have no simultaneous solution.

The finding of a solution to simultaneous equations is one of *the* fundamental problem solving techniques of mathematics. Very many mathematical problems require you, at some stage, to solve simultaneous equations.