Simultaneous equations, and the intersection of lines and curves

Simultaneous equations where one equation is quadratic

Two linear simultaneous equations, of the form

$ax + by = c$

$dx + ey = f$

can be solved using the methods of elimination, substitution, or by graphical representation. In this chapter we learn that it is also possible to solve simultaneous equations where one of the equations is quadratic. In the following example

$x - 2y = 3$

$x^2 - 3y^2 = 22$

the second equation is quadratic. Solving the equations means finding values of x and y that simultaneously satisfy the two equations.

Method of substitution

The aim of this method is to use one equation to express one of the variables solely in terms of the other and then to substitute this expression into the remaining equation.

$x - 2y = 3$

$x^2 - 3y^2 = 22$

Rearranging the first equation

$x = 3 + 2y$

Substituting into the second equation

$(3 + 2y)^2 - 3y^2 = 22$

$9 + 12y + 4y^2 - 3y^2 = 22$

$y^2 + 12y - 13 = 0$

Solving the quadratic

$(y - 1)(y + 13) = 0$

$y = 1 \text{ or } -13$

We have two solutions for y, which is to be expected as we are dealing with a quadratic equation. This will yield two solutions for x as we substitute each of the y values back into $x = 3 + 2y$.

Substituting $y = 1$ gives

$x = 3 + 2 = 5$

Substituting $y = -13$ gives

$x = 3 + 2(-13) = 3 - 26 = -23$

We now have the complete solutions to our pair of simultaneous equations.

$y = 1$ and $x = 5$ and $y = -13$ and $x = -23$

These pairs of x and y values cannot be swapped around, for example it would be incorrect to state: $y = 1$ and $x = -23$. One must specify which x and y values go together

Graphical Representation

Consider the two equations

$2y - x = -2$

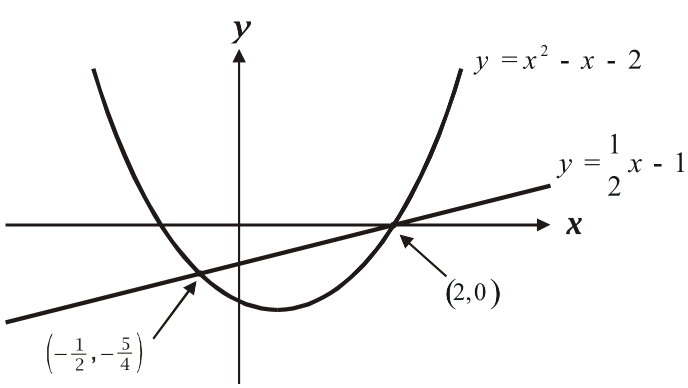
$y = x^2 - x - 2$

Let us rearrange the first equation into the form $y = mx + c$, the equation of a straight line.

$2y - x = -2$

$y = \frac{1}{2}x - 1$

This allows us to plot the two equations on a graph.



The two solutions to the simultaneous equations are obtained simply by reading off the coordinates of the points of intersection of the two lines.

$y = -\frac{5}{4}$ and $x = -\frac{1}{2}$ and $y = 0$ and $x = 2$

It can now be seen that it is important to keep the pairs of x and y values together, as they represent the coordinates of the points of intersection.

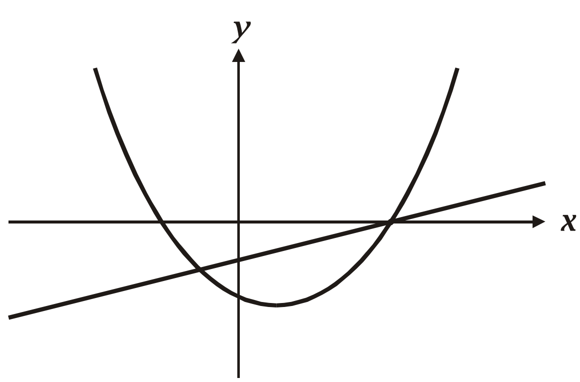
Intersection of a line and a curve

As the previous example illustrates, sets of equations such as

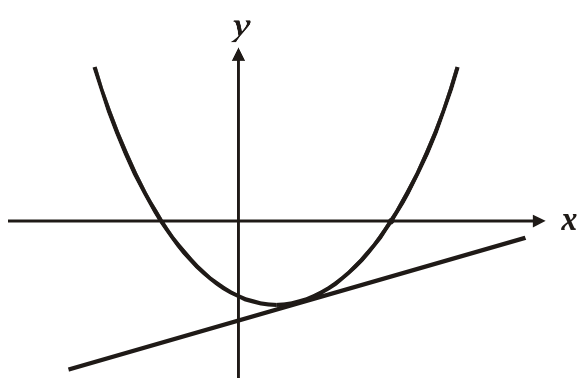
$2y - x = -2$ $\qquad\qquad$ $y = x^2 - x - 2$

represent the intersection of a line and a curve. We wish now to be able to consider the general case concerning the intersection of a line and a curve. The general point about the intersection of a line and a curve is as follows: either the line meets the curve at one or more distinct points of intersection; and/or the line is tangent to the curve at one or more points, or the line does not meet the curve at all.

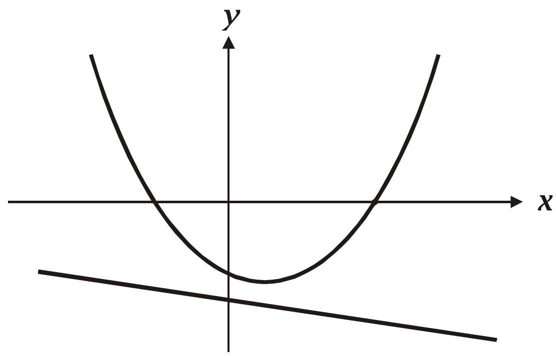
(1) All points of intersection are distinct.



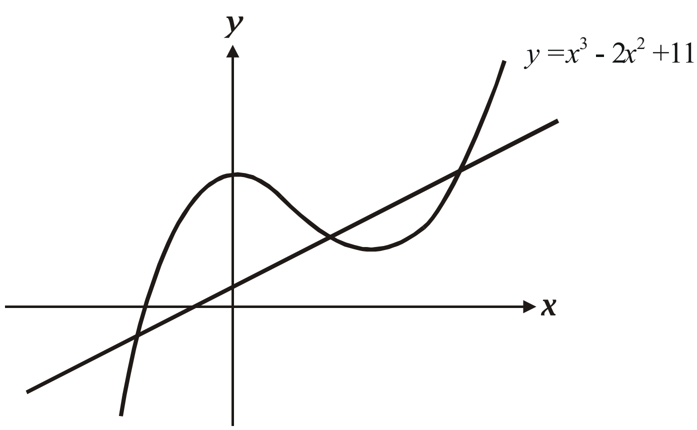
(2) A line is tangent to a curve when it just touches it. When a line and a curve intersect, the line may be tangent to the curve at one or possibly more points.



(3) The line and the curve do not meet.



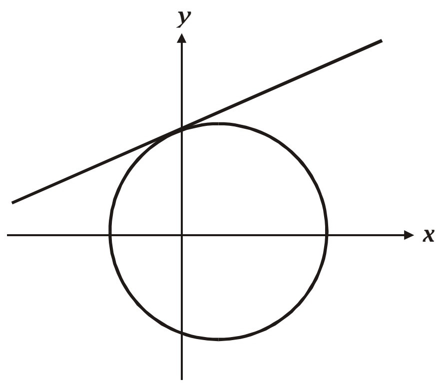
Thus, when solving simultaneous equations, where one equation represents a line and the other a curve, there need not always be a solution. The maximum number of solutions corresponds to the order of the polynomial representing the curve. For example, the order of $y = x^3 - 2x^2 + 11$ is three, since the highest term in x in the equation is x^3; in this case any line can intersect with $y = x^3 - 2x^2 + 11$ in at most three places.



Algebraically, the intersection of a line and a curve is solved by the method of substitution.

Tangents

In certain cases we can use these geometric intuitions to prove that a line is tangent to a curve. Recall that a line is tangent to a curve when it just touches it. Consider a line and a circle.



If the equations of a line and a circle have just one solution, then the line must touch the circle and consequently be tangent to it.

Example

Show that the line $y = \frac{1}{\sqrt{3}}x + \sqrt{3}$ is tangent to the circle $(x-1)^2 + y^2 = 4$ and find the point where it meets this circle. Sketch the circle and the line.

Solution

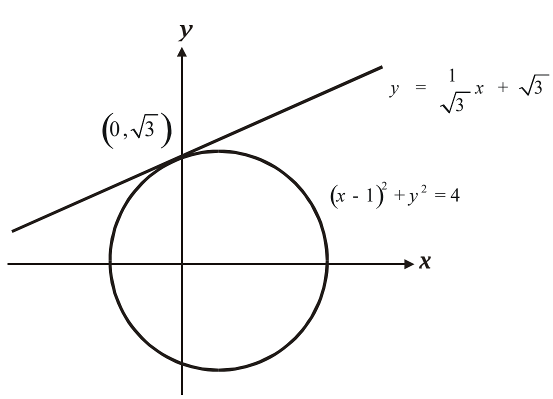
Substituting $y = \frac{1}{\sqrt{3}}x + \sqrt{3}$ in $(x-1)^2 + y^2 = 4$ we obtain

$$(x-1)^2 + \left(\frac{x}{\sqrt{3}} + \sqrt{3}\right)^2 = 4$$

$$x^2 - 2x + 1 + \frac{x^2}{3} + 2x + 3 = 4$$

$$\frac{4x^2}{3} = 0 \quad \Rightarrow \quad x = 0$$

There is only one solution, so the line must just touch the circle once, and is consequently tangent to it. When $x = 0$, $y^2 = 3$, hence $y = \sqrt{3}$. A sketch of the circle and the line is



Intersection of two curves

Likewise, curves intersect with curves either at distinct points, at tangents, or not at all. However, the intersection of curves can only rarely be solved by the method of substitution.

Example

Find the points of intersection of the circle $x^2 + (y-1)^2 = 18$ and the parabola $y = x^2 - 5$

Solution

It is possible to solve this pair of simultaneous equations, because x^2 can be eliminated from both. We begin by making x^2 the subject of the second equation

$x^2 = y + 5$

Then on substituting this into the equation for the circle we obtain

$(y+5) + (y-1)^2 = 18$

$y + 5 + y^2 - 2y + 1 = 18$

$y^2 - y - 12 = 0$

$(y-4)(y+3) = 0$

$y = 4$ or $y = -3$

When $y = 4$, $x^2 = 9$, hence $x = \pm 3$

When $y = -3$, $x^2 = 2$, hence $x = \pm\sqrt{2}$