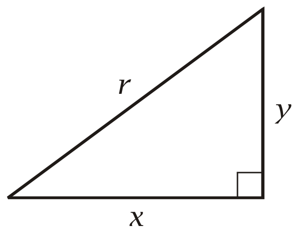
Sine and cosine rules

Pythagoras's theorem

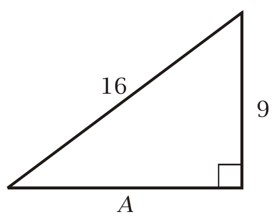
You should already be familiar with Pythagoras's theorem, which states that *in a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares of the other two sides.*



$r^2 = x^2 + y^2$

Example (1)

Find the length A, in the following triangle, giving your answer in surd form.

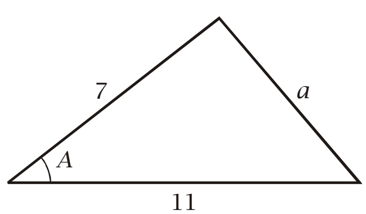


Solution

$16^2 = a^2 + 9^2$

$a = \sqrt{16^2 - 9^2} = \sqrt{175} = 5\sqrt{7}$

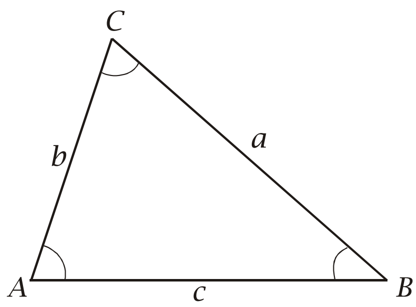
We want to generalise Pythagoras's Theorem to include any triangle.



Although this is not a right-angled triangle, we would still like to be able to find an unknown length or angle. The diagram suggests that if the angle A is known then we can find the length a, or vice-versa, if the length a is known then we can find the angle A.

The cosine formula

The *cosine formula* is a generalised form of Pythagoras's Theorem to deal with triangles where there is no right angle. In the following diagram the angles are labelled *A, B, C* and sides opposite those angles are labelled *a, b, c* respectively.

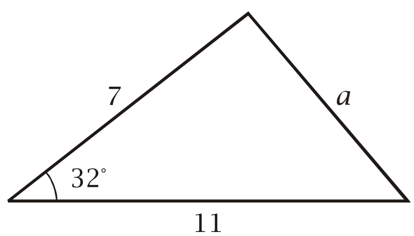


Using these labels the cosine formula is

$a^2 = b^2 + c^2 - 2bc\cos A$

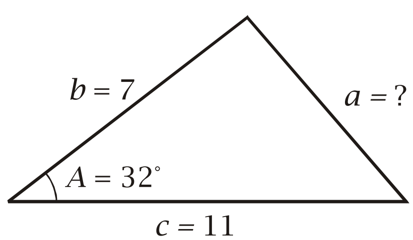
Example (2)

In the following triangle, find *a*.



Solution

In the cosine formula, $a^2 = b^2 + c^2 - 2bc\cos A$, the symbol A stands for the *included* angle and the symbols *b* and *c* stand for the sides enclosing the angle *A*. The unknown side is opposite *A* and is labelled *a*.



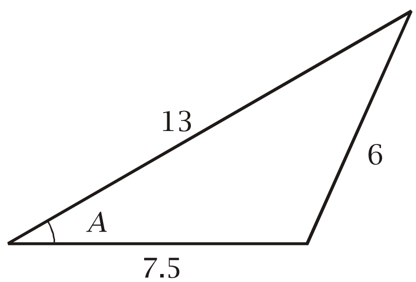
Substituting $b = 7$, $c = 11$ and $A = 32°$ into $a^2 = b^2 + c^2 - 2bc\cos A$

$$
\begin{aligned}
a^2 &= 7^2 + 11^2 - 2 \times 7 \times 11 \times \cos 32 \\
&= 49 + 121 - 154 \times 0.848... \\
&= 39.400...
\end{aligned}
$$

$$a = \sqrt{39.400} = 6.2769... = 6.28 \ (3.\text{s.f.})$$

If we know three sides in a triangle we can find any one of the unknown angles.

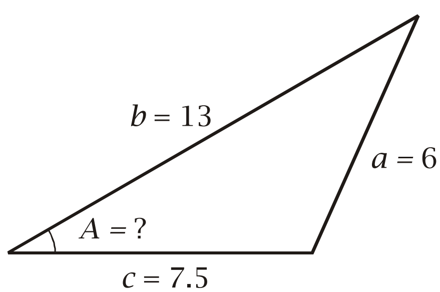
Example (3)



In the diagram above, find the angle A.

Solution

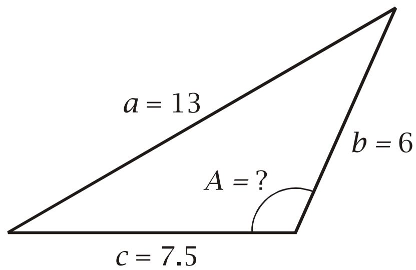
In the cosine formula $a^2 = b^2 + c^2 - 2bc\cos A$ if A is the included angle, b and c are the sides that enclose it.



Substituting into $a^2 = b^2 + c^2 - 2bc\cos A$ gives

$$
\begin{aligned}
&6^2 = 13^2 + 7.5^2 - 2 \times 13 \times 7.5 \times \cos A \\
&2 \times 13 \times 7.5 \times \cos A = 13^2 + 7.5^2 - 6^2 \\
&195 \cos A = 189.25 \\
&A = \cos^{-1}\left(\frac{189.25}{195}\right) = 13.948.... = 13.9^\circ \ (0.1^\circ)
\end{aligned}
$$

In the formula $a^2 = b^2 + c^2 - 2bc\cos A$ the labels a, b, c and A are relative – they move around depending on what you are trying to find. For example, suppose in the same triangle we are asked to find the angle between the sides of length 6 and 7.5, then we would label it as follows.



The positions of b and c can be swapped; they stand for the two sides that enclose the angle in question in any order.

Example (4)

Find the angle A in the diagram above.

Solution

Substituting into $a^2 = b^2 + c^2 - 2bc\cos A$

$$13^2 = 6^2 + 7.5^2 - 2 \times 6 \times 7.5 \times \cos A$$

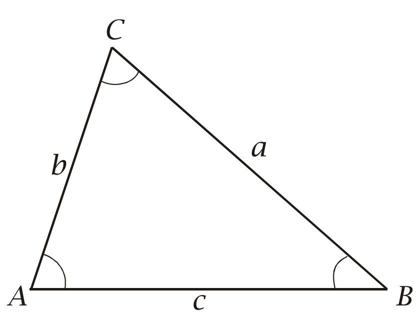
$$\cos A = \frac{6^2 + 7.5^2 - 13^2}{2 \times 6 \times 7.5} = -0.8577...$$

$$A = \cos^{-1}(-0.8577) = 148.515... = 148.5^\circ \quad (0.1^\circ)$$

The formula still works despite the negative value of $\cos A$ because the calculator finds a value for the inverse of cosine, $\cos^{-1}$, that lies between 0 and 180°.

The sine rule

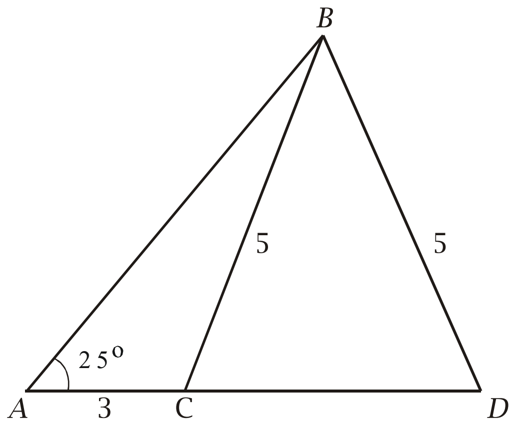
Later we shall prove the cosine formula. Before we do so we shall introduce another formula used in triangles. This is the *sine rule*. As before, we label the triangle thus



The sine rule is

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example (5)



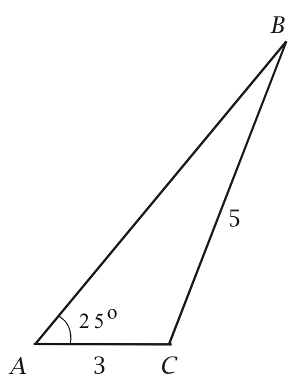
The diagram shows four points in a plane with lengths and angles as marked.

(*a*) Find the angle $A\hat{B}C$ to 0.1°.

(*b*) Find the distance *CD* to two significant figures.

Solution

(*a*)



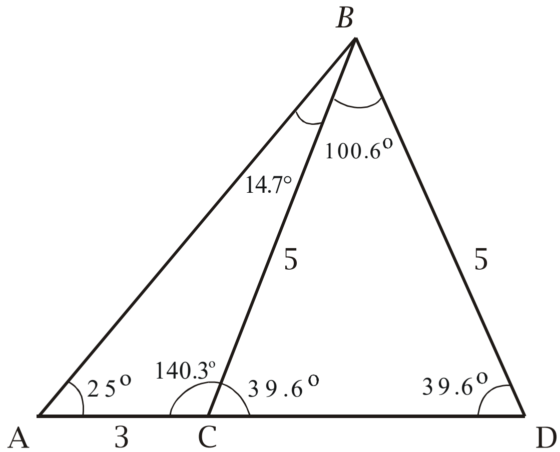
On substituting into the sine rule $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\frac{\sin B}{3} = \frac{\sin 25}{5}$$

$$\sin B = \frac{3}{5}\sin 25$$

$$A\hat{B}C = B = \sin^{-1}(0.2535...) = 14.7° (\text{to } 0.1°)$$

(*b*) Since $B = 14.7°$ we can fill in some of the missing angles in the diagram.



The triangle in question is *BCD* and on substituting $A = 100.6°$, $b = c = 5$ into the cosine formula $a^2 = b^2 + c^2 - 2bc\cos A$

$$CD^2 = 5^2 + 5^2 - 2 \times 5 \times 5 \times \cos 100.6 = 59.19756....$$
$$CD = 7.6939... = 7.7\,(2\text{s.f.})$$

Area of a triangle

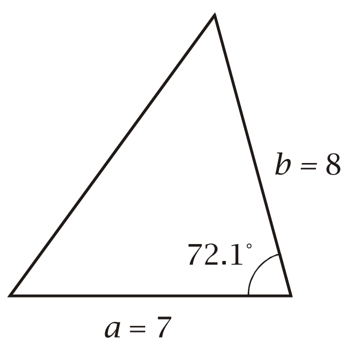
There is a third formula of importance, this is a formula for the area of a triangle, Δ, given by

$$\Delta = \frac{1}{2}ab\sin C$$

where *C* is the angle enclosed by the sides *a* and *b*.

Example (6)

Find the area of this triangle



Solution

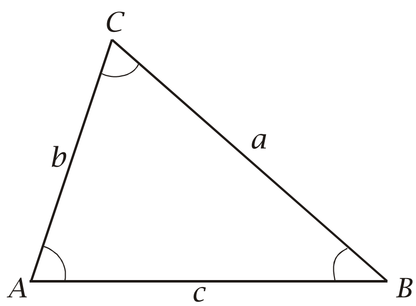
Substituting into $\Delta = \frac{1}{2}ab\sin C$

$$\Delta = \frac{1}{2}7\times 8\times \sin 72.1 = 26.6446... = 26.6 \text{ sq. units} \quad (3.s.f.)$$

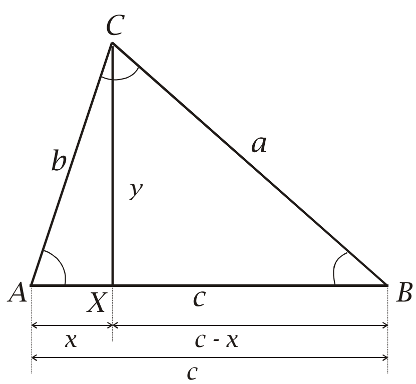
Proofs

(1) Proof of the cosine formula

Consider the triangle ABC



Drop a perpendicular from C to AB as follows.



In the ΔAXC from Pythagoras's theorem

$y^2 = b^2 - x^2$

In the ΔCXB also from Pythagoras's theorem

$y^2 = a^2 - (c - x)^2$

Hence, eliminating y from these two equations gives]

$b^2 - x^2 = a^2 - (c - x)^2$

$b^2 - x^2 = a^2 - (c^2 - 2cx + x^2)$

$b^2 - x^2 = a^2 - c^2 + 2cx - x^2$

Rearranging gives

$a^2 = b^2 + c^2 - 2cx$

In the ΔAXC

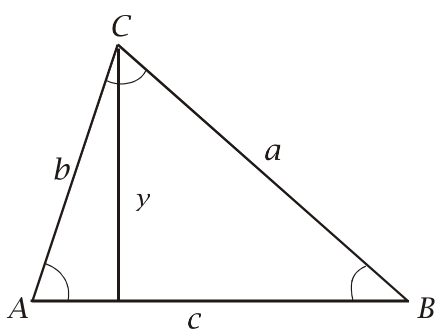
$x = b\cos A$

Hence

$a^2 = b^2 + c^2 - 2bc\cos A$

(2) Proof of the sine formula

Consider the triangle *ABC*, with a perpendicular from *C* to *AB* as follows



In the ΔACX

$y = b\sin A$

In the ΔCXB

$y = a\sin B$

Eliminating y from both equations gives

$b\sin A = a\sin B$

or

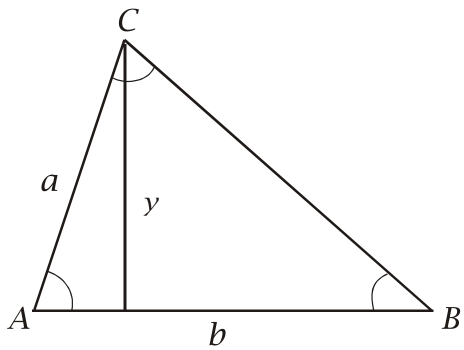
$\frac{\sin A}{a} = \frac{\sin B}{b}$

The argument could be repeated for any two pairs of angles, hence]

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

(3) Area of a triangle

Consider the triangle *ABC*, with perpendicular from *A* to *CB* as follows



The area of the triangle is given by

$\Delta = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}a \times y$

But

$y = b\sin C$

Hence

$\Delta = \frac{1}{2}ab\sin C$