Sinusoidal functions

Prerequisites

You should be familiar with the following compound-angle formulae

(1) $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$

(2) $\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$

(3) $\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$

(4) $\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$

Example (1)

By substitution into one of the compound-angle formulae prove

(a) $\sin\theta \equiv \cos\left(\theta - \frac{\pi}{2}\right)$

(b) $\cos\theta \equiv \sin\left(\theta + \frac{\pi}{2}\right)$

Solution

(a) To prove

$$\sin\theta \equiv \cos\left(\theta - \frac{\pi}{2}\right)$$

The appropriate compound-angle formula is

$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

Substituting $A = \theta$ and $B = \frac{\pi}{2}$

$$\cos\left(\theta - \frac{\pi}{2}\right) \equiv \cos\theta\cos\frac{\pi}{2} + \sin\theta\sin\frac{\pi}{2}$$

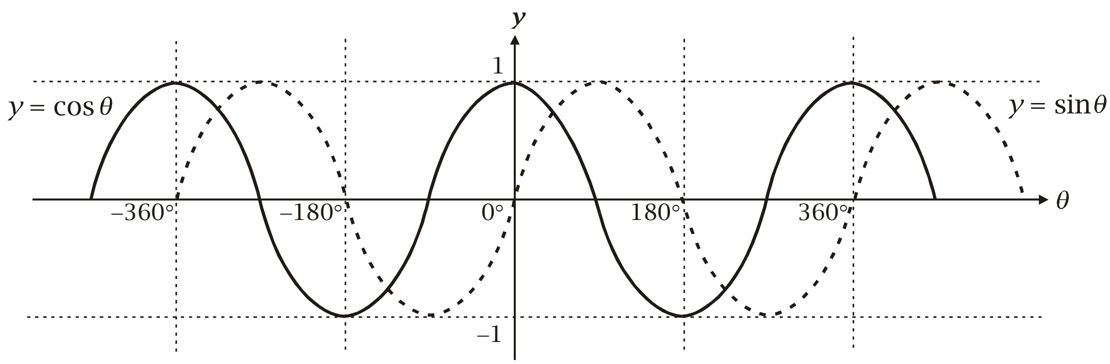
$$\equiv \sin\theta \quad \left[\text{since } \cos\frac{\pi}{2} = 0 \text{ and } \sin\frac{\pi}{2} = 1\right]$$

(b) $\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$

Let $A = \theta$ and $B = \frac{\pi}{2}$

$$\sin\left(\theta + \frac{\pi}{2}\right) \equiv \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2} \equiv \cos\theta$$

What this first example shows is that $\sin\theta$ can be written in terms of $\cos\theta$ and vice-versa. This can be illustrated from a graph superimposing $\sin\theta$ on $\cos\theta$.



So $\cos\theta$ arises from $\sin\theta$ by a translation along the θ-axis by $-\frac{\pi}{2}$, and $\cos\theta \equiv \sin\left(\theta + \frac{\pi}{2}\right)$. Also $\sin\theta$ arises from $\cos\theta$ by a translation along the θ-axis by $+\frac{\pi}{2}$, and $\sin\theta \equiv \cos\left(\theta - \frac{\pi}{2}\right)$. In these cases the angle $\frac{\pi}{2}$ is called the *phase shift* – that is, $\cos\theta$ is the phase shift by $-\frac{\pi}{2}$ of $\sin\theta$; $\sin\theta$ is the phase shift by $+\frac{\pi}{2}$ of $\cos\theta$.

Sinusoidal functions

Sinusoidal functions are linear combinations of sine and cosine functions of the same variable and with the same period. In other words they are functions of the form

$f(\theta) = a\cos\theta - b\sin\theta$

where a and b are real numbers. Sinusoidal functions are equivalent to a single sine or cosine function shifted along the θ-axis by phase shift α. It is consequently possible to combine $f(\theta) = a\cos\theta - b\sin\theta$ into a single function either of the form $f(\theta) = R\cos(\theta + \alpha)$ or of the form $f(\theta) = R\sin(\theta + \alpha')$ where α and α' are the appropriate phase shift of the cosine and sine functions respectively and R is the *amplitude* of $f(\theta)$. Suppose we are given $f(\theta) = a\cos\theta - b\sin\theta$. Then let

$a\cos\theta - b\sin\theta \equiv R\cos(\theta + \alpha)$

where R and α are real numbers. By the compound angle formula

$R\cos(\theta+\alpha) \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Hence

$a\cos\theta - b\sin\theta \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Since this is an identity the left-hand side is equivalent to the right-hand side for all values of θ.

Substituting $\theta = 0 \quad \Rightarrow \quad \sin\theta = 0, \ \cos\theta = 1$

gives $a = R\cos\alpha$. Substituting $\theta = \frac{\pi}{2} \quad \Rightarrow \quad \sin\theta = 1, \ \cos\theta = 0$

gives $b = R\sin\alpha$. From this it follows that

$$R = \sqrt{a^2 + b^2} \qquad \alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

Alternatively, writing

$a\cos\theta - b\sin\theta \equiv R\sin(\theta + \alpha')$

and substituting

$a\cos\theta - b\sin\theta \equiv R\sin(\theta+\alpha') = R\sin\theta\cos\alpha' + R\cos\theta\sin\alpha'$

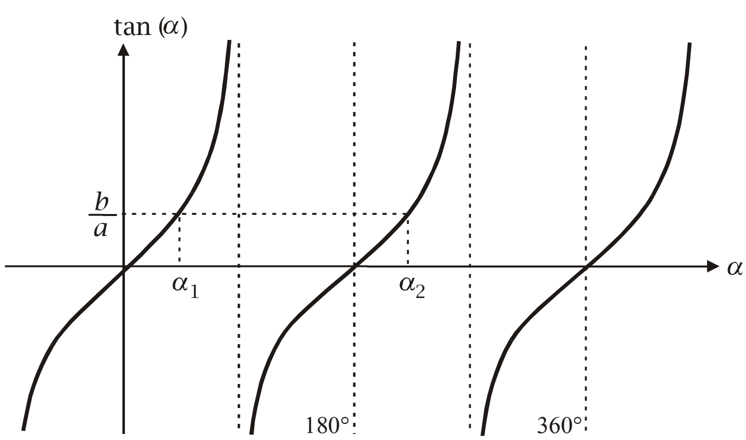
$R\cos\alpha' = -b \qquad R\sin\alpha' = a$

$$R = \sqrt{a^2 + b^2} \qquad \tan\alpha' = -\frac{a}{b}$$

A small problem and its resolution

The appropriate technique here is given $f(\theta) = a\sin\theta + b\cos\theta$ to find R and α such that $f(\theta) = R\cos(\theta + \alpha)$.

This requires solution of $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$. However, the function $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$ is periodic and has two solutions in the domain 0 to 2π.



Let us denote the two solutions of $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$ in the domain $0° \le \alpha \le 360°$ by α_1 and α_2. The calculator will find a value of $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$ that lies in the principle domain $-90° \le \alpha \le 90°$, so this solution may not even lie in the domain $0° \le \alpha \le 360°$. As only one of α_1 and α_2 is the correct phase shift of $f(\theta) = a\sin\theta + b\cos\theta$, we require a further technique to determine the right choice of phase shift. This difficulty can be circumvented if we draw a diagram showing the signs of a and b and deduce from it the correct phase angle.

Worked example (2)

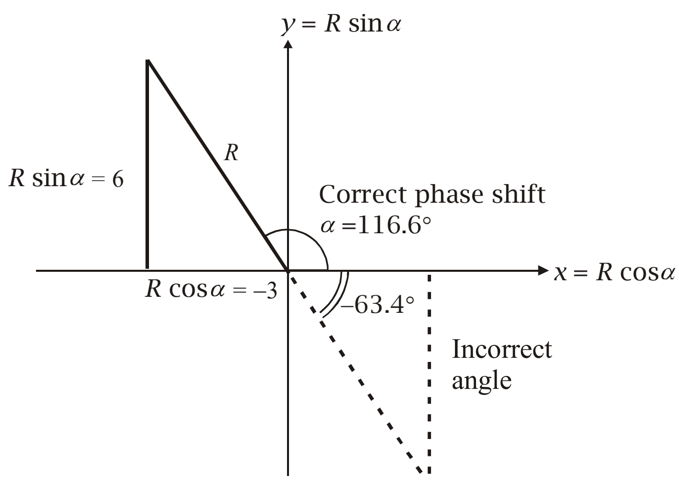
We are asked to write $6\cos x - 3\sin x$ as a single sinusoidal function of the form $R\sin(\theta + \alpha)$. Then we begin by writing

$$6\cos x - 3\sin x = R\sin(x + \alpha)$$
$$= R\sin x\cos\alpha + R\cos x\sin\alpha$$

Hence

$$R\sin\alpha = 6$$
$$R\cos\alpha = -3$$
$$R = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$$
$$\alpha = \tan^{-1}\left(\frac{6}{-3}\right) = -63.4° \pm 180°n \qquad n = 0, \pm1, \pm2...$$

At this point a diagram is sketched to determine the correct phase shift.



The diagram shows how the value of $x = R\cos\alpha$ and $y = R\cos\alpha$ varies with changes in the phase shift α.

Both 116.6° and $-63.4° \equiv 296.6°$ have $\tan\left(-\frac{6}{3}\right)$. However, in this question the correct phase shift must give

$R\sin\alpha = 6$
$R\cos\alpha = -3$

so the correct value of α is 116.6°. Thus

$$6\cos x - 3\sin x = 3\sqrt{5}\sin\left(x + 116.6°\right) \qquad (\text{nearest } 0.1°).$$

Example (3)

Find all the values of θ in the range $0° \le \theta \le 360°$ satisfying

$$\cos\theta - 3\sin\theta = \sqrt{5}$$

giving your answer in degrees correct to one decimal place.

Solution

Let

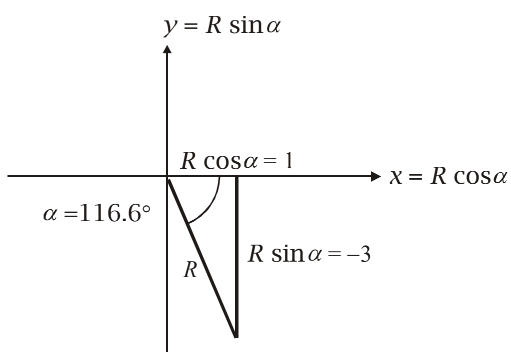
$$\cos\theta - 3\sin\theta = R\cos\left(\theta + \alpha\right) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

Hence

$R\cos\alpha = 1$
$R\sin\alpha = -3$

$$R = \sqrt{1^2 + \left(-3\right)^2} = \sqrt{10}$$

$$\alpha = \tan^{-1}\left(\frac{-3}{1}\right)$$



From the diagram

$$\alpha = \tan^{-1}\left(\frac{-3}{1}\right) = -71.6° \pm n180° \qquad n = 1, 2, 3, \ldots$$

Thus $\cos\theta - 3\sin\theta = \sqrt{5}$ implies

$\sqrt{10}\cos(\theta - 71.6°) = \sqrt{5}$

$\cos(\theta - 71.6°) = \frac{1}{\sqrt{2}}$

$\theta - 71.6° = \ldots -45°, 45°, 315°, \ldots$

$\theta = 26.6°, 116.6°$ where $0° \le \theta \le 360°$ (nearest 0.1°)

Relationship between a function and its reciprocal

Let $y = f(x)$, then the reciprocal of this function is $\frac{1}{y} = \frac{1}{f(x)}$. Suppose further that the function $y = f(x)$ is always positive, $f(x) > 0$ for all values of *x*. Then $f(x) \neq 0$ for any *x* and the reciprocal $\frac{1}{y} = \frac{1}{f(x)}$ is defined for all values of *x* in the domain of $f(x)$. Then clearly as $y = f(x)$ gets larger and larger then $\frac{1}{y} = \frac{1}{f(x)}$ gets smaller and smaller, and vice-versa. So a maximum of $y = f(x)$ corresponds to a minimum of $\frac{1}{y} = \frac{1}{f(x)}$, and a minimum of $y = f(x)$ corresponds to a maximum of $\frac{1}{y} = \frac{1}{f(x)}$

Example (4)

(*a*) Express $12\cos x - 5\sin x$ in the form $R\cos(x + \alpha)$ were R and α are constants with $R > 0$ and $0° < \alpha < 90°$.

(*b*) Hence find the greatest value of

$$\frac{1}{12\cos x - 5\sin x + 20}$$

Solution

(*a*) $12\cos x - 5\sin x = R\cos(x + \alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$

Hence

$R\cos\alpha = 12$

$R\sin\alpha = 5$

$R = \sqrt{12^2 + 5^2} = 13$

$\alpha = \tan^{-1}\left(\frac{5}{12}\right) = 22.6°$ (nearest 0.1°)

Note in this solution because both

$R\cos\alpha = 12$
$R\sin\alpha = 5$

are positive magnitudes the phase angle must lie in the first quadrant. The question also specified $0° < \alpha < 90°$. Then

$12\cos x - 5\sin x = 13\cos(x + 22.6°)$

(*b*) The minimum value of

$f(x) = 12\cos x - 5\sin x = 13\cos(x + 22.6°)$

is -13 when $\cos(x + 22.6°) = -1$ [that is when $x = 202.6° + n360°$].

Then $f(x) + 20 = 12\cos x - 5\sin x + 20$ has minimum value $20 - 13 = 7$. So the greatest value of $\frac{1}{12\cos x - 5\sin x + 20}$ is $\frac{1}{7}$.